On quantum dissipative systems: ground states and orbital stability

Thierry Goudon, Léo Vivion

On quantum dissipative systems: ground states and orbital stability

Thierry Goudon*[1] and Léo Vivion†[1]

[1]Université Côte d'Azur, Inria, CNRS, LJAD,
Parc Valrose, F-06108 Nice, France

Abstract

We investigate the existence and stability of ground states for a model coupling the Schrödinger equation to the wave equation in transverse directions. The model is intended to describe complex interactions between quantum particles and their environment. The result can be interpreted as a dissipation statement, induced by the energy exchanges with the environment. The proofs use either concentration-compactness arguments or spectral analysis of the linearized energy. Difficulties arise related to the fact the model does not satisfy scale invariance properties.

Keywords. Open quantum systems. Particles interacting with a vibrational field. Schrödinger-Wave equation. Ground states. Orbital stability.

Math. Subject Classification. 35Q40 35Q51 35Q55

1 Introduction

This paper is concerned with the study of the following system of PDEs, hereafter referred to as the *Schrödinger-Wave equation*

$$\left(i\partial_t u + \frac{1}{2}\Delta_x u\right)(t,x) = \left(\int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(x-y)\sigma_2(z)\psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z\right)u(t,x), \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d \tag{1a}$$

$$(\partial^2_{tt}\psi - c^2\Delta_z\psi)(t,x,z) = -c^2\sigma_2(z)\left(\int_{\mathbb{R}^d}\sigma_1(x-y)|u(t,y)|^2\,\mathrm{d}y\right), \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d,\ z\in\mathbb{R}^n \tag{1b}$$

*thierry.goudon@inria.fr
†leo.vivion@unice.fr

endowed with the initial data

$$u(0,x) = u_0(x), \qquad (\psi(0,x,z), \partial_t\psi(0,x,z)) = (\psi_0(x,z), \psi_1(x,z)). \tag{2}$$

Here u represents the wave function of a quantum particle, which interacts with the vibrational field ψ, and $c > 0$ is a fixed parameter. A key feature of the model is the fact that the particle motion holds in the space $\mathbb{R}^d$, but the vibrations hold in a *transverse direction* $\mathbb{R}^n$. We are mainly interested in finding particular *solitary wave* solutions of the system, with the specific form

$$u(t,x) = e^{i\omega t}Q(x), \qquad \psi(t,x,z) = \Psi(x,z) \tag{3}$$

where $\omega \in \mathbb{R}$, and Q, Ψ are real valued, and to investigate the stability of such solutions.

1.1 Motivation

This work is motivated by the modeling of dissipative systems. As suggested by A. Caldeira and A. Legget [4] the dissipation arising on a physical system might come from a coupling with a complex environment. In this approach, dissipation is interpreted as the transfer of energy from the single degree of freedom characterising the system to the more complex set of degrees of freedom describing the environment; the energy is then evacuated into the environment and does not come back to the system. There are many possible descriptions of the environment: the case in which the environmental variables are vibrational degrees of freedom is particularly appealing. The system (1a)-(1b) belongs to this class of models.

This system is nothing but a quantum version of a model introduced by L. Bruneau and S. de Bièvre in [3] for describing a classical particle interacting with its environment seen as a bath of oscillators. Roughly speaking in each space position $x \in \mathbb{R}^d$ there is a membrane oscillating on a transverse direction $z \in \mathbb{R}^n$. When the particle hits a membrane, its kinetic energy activates vibrations and the energy is evacuated at infinity in the $\mathbb{R}^n$ directions. In particular, the coordinates $(z_1, ..., z_n) \in \mathbb{R}^n$ need not have the specific dimension of a length (but adopting this language might definitely help the intuition). These energy transfer mechanisms eventually act as a sort of friction force on the particle, an intuition rigorously justified in [3, Theorem 2 and Theorem 4]. The system for the position of the particle $t \mapsto q(t)$ and the state of the vibrational environment $(t,z) \mapsto \psi(t,z)$ reads

$$\ddot{q}(t) = -\int \nabla\sigma_1(q(t)-y)\sigma_2(z)\psi(t,y,z)\,\mathrm{d}z\,\mathrm{d}y, \qquad t \in \mathbb{R} \tag{4a}$$

$$(\partial_{tt}^2\psi - c^2\Delta_z\psi)(t,z) = -\sigma_2(z)\,\sigma_1(x-q(t)), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d,\ z \in \mathbb{R}^n \tag{4b}$$

completed by the initial data

$$(q(0), \dot{q}(0)) = (q_0, p_0), \qquad (\psi(0,x,z), \partial_t\psi(0,x,z)) = (\psi_0(x,z), \psi_1(x,z)). \tag{5}$$

The functions $\sigma_1 : \mathbb{R}^d \to [0,\infty)$ and $\sigma_2 : \mathbb{R}^n \to [0,\infty)$ are form functions encoding the interaction domain between the particle and the environment. The model can be extended by considering P-interacting particles, and the mean-field regime $P \to \infty$ leads to the following Vlasov-Wave system

[12]

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\left(\sigma_1 \star_x \int \sigma_2\psi\,\mathrm{d}z\right)\cdot\nabla_v f = 0, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d \tag{6a}$$

$$\partial^2_{tt}\psi - c^2\Delta_z\psi = -\sigma_2(z)\left(\sigma_1\star_x\int f\,\mathrm{d}v\right), \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d,\ z\in\mathbb{R}^n \tag{6b}$$

$$f(0,x,v) = f_0(x,v), \qquad (\psi(0,x,z),\partial_t\psi(0,x,z)) = (\psi_0(x,z),\psi_1(x,z)), \tag{6c}$$

where f stands for the particle distribution function in phase space. This system is thoroughly investigated in [1, 9, 38]. In [8], it is proposed to rescale the wave equation (6b) as follows

$$\partial^2_{tt}\psi - c^2\Delta_z\psi = -c^2\sigma_2\left(\sigma_1\star_x\int f\,\mathrm{d}v\right). \tag{7}$$

As c goes to $+\infty$, the solutions of the rescaled system (6a), (7) tend to solutions of

$$\partial_t\tilde f + v\cdot\nabla_x\tilde f - \nabla_x\left(\sigma_1\star_x\int\sigma_2\tilde\psi\,\mathrm{d}z\right)\cdot\nabla_v\tilde f = 0, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d \tag{8a}$$

$$-\Delta_z\tilde\psi = -\sigma_2\left(\sigma_1\star_x\int\tilde f\,\mathrm{d}v\right), \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d,\ z\in\mathbb{R}^n \tag{8b}$$

(Without the rescaling the regime $c\to\infty$ would simply lead to the free transport equation for the particle distribution function $\tilde f$.) We can write

$$\tilde\psi(t,x,z) = \Gamma(z)\,\left(\sigma_1\star\int\tilde f\,\mathrm{d}v\right)(x)$$

where Γ denotes the unique solution of

$$-\Delta_z\Gamma = -\sigma_2, \qquad \Gamma\in H^1(\mathbb{R}^n_z). \tag{9}$$

This observation allows us to express (8a)-(8b) as a standard Vlasov equation

$$\partial_t\tilde f + v\cdot\nabla_x\tilde f + \kappa\nabla_x\left(\Sigma\star_x\int\tilde f\,\mathrm{d}v\right)\cdot\nabla_v\tilde f = 0, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \tag{10}$$

where the potential is defined by a convolution with the macroscopic density, with

$$\kappa = \|\nabla_z\Gamma\|^2_{L^2_z}, \qquad \Sigma = \sigma_1\star\sigma_1. \tag{11}$$

Quite surprisingly – mind the sign $\kappa > 0$ – this corresponds to an attractive dynamics. This unexpected connection guides the intuition to establish further features of the solutions of the Vlasov-Wave system; it particular, they exhibit Landau damping phenomena [13, 14]. The analysis of these models, either for a single particle or the kinetic description, brings out the critical role of the wave speed $c > 0$ and the dimension n of the space for the wave equation.

The system (1a)-(1b) then appears as the quantum version of the L. Bruneau and S. de Bièvre model. This intuition can be justified by the semi-classical analysis *à la* P.-L. Lions-T. Paul [26], which makes a natural connection between the Vlasov-Wave system and (1a)-(1b), see Appendix B and [39]. Note that here we have adopted from the beginning the rescaling where the coupling term in the wave equation (1b) is of the order of c^2. We will motivate this choice below. According to the framework introduced in [3], throughout this article we assume:

(H1) $n \geq 3$,

(H2) The form functions σ_1 and σ_2 are non-negative, smooth, compactly supported and radially symmetric.

As said above the role of the dimension n for the wave equation is critical in these models. Indeed, the evacuation of energy in the environment relies on the dispersion properties of the wave equation, which are strong enough when n is sufficiently large [13]. By the way, notice that the definition of κ in (11) makes sense when assuming $n \geq 3$. The case $n = 3$ also plays a specific role in the theory presented in [3]. The assumptions **(H1)** and **(H2)** on the form functions are very natural in the modeling framework of [3]. In what follows, we use the abuse of notation to mix up a radially symmetric function of $x \in \mathbb{R}^d$ with the underlying function of the scalar quantity $|x|$, and we will equally refer to the monotonicity of this function. Following the observations made for classical particles, it is instructive to consider the regime where c goes to $+\infty$ in (1a)–(1b). We are led to

$$i\partial_t \tilde{u} + \frac{1}{2}\Delta_x \tilde{u} = \left(\sigma_1 \star_x \int \sigma_2 \tilde{\psi}\,\mathrm{d}z\right)\tilde{u}, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d, \tag{12a}$$

$$-\Delta_z \tilde{\psi} = -\sigma_2(z)\left(\sigma_1 \star_x |\tilde{u}|^2\right)(x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d,\ z \in \mathbb{R}^n \tag{12b}$$

which can be cast in the usual form of an Hartree type equation

$$i\partial_t \tilde{u} + \frac{1}{2}\Delta_x \tilde{u} = -\kappa\left(\Sigma \star_x |\tilde{u}|^2\right)\tilde{u}, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d. \tag{13}$$

This remark will be helpful for the analysis.

The conservation of the total energy is a remarkable property of all these models. For the particle equation (4a)-(4b), we set

$$\mathcal{E}_{\mathrm{part}}(t) = \frac{\dot{q}(t)}{2} + \frac{1}{2}\int (|\partial_t\psi|^2 + c^2|\nabla_z\psi|^2)(t,x,z)\,\mathrm{d}z\,\mathrm{d}x + \int \sigma_1(q(t)-y)\sigma_2(z)\psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z$$

and for for the kinetic equation (6a), with (7) (mind the rescaling for the wave equation), we set

$$\begin{aligned}\mathcal{E}_{\mathrm{kin}}(t) \quad &= \quad \frac{1}{2}\int v^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int \Big(\frac{|\partial_t\psi|^2}{c^2} + |\nabla_z\psi|^2\Big)(t,x,z)\,\mathrm{d}z\,\mathrm{d}x \\ &\quad + \int \sigma_1(x-y)\sigma_2(z)\psi(t,y,z)f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.\end{aligned}$$

Then, we have

$$\mathcal{E}_{\mathrm{part}}(t) = \mathcal{E}_{\mathrm{part}}(0), \qquad \mathcal{E}_{\mathrm{kin}}(t) = \mathcal{E}_{\mathrm{kin}}(0).$$

For the quantum model, (1a)–(1b), it becomes

$$\begin{aligned}\mathcal{E}_{\mathrm{Schr}}(t) \quad &= \quad \frac{1}{2}\int |\nabla_x u(t,x)|^2\,\mathrm{d}x + \frac{1}{2}\int \Big(\frac{|\partial_t\psi|^2}{c^2} + |\nabla_z\psi|^2\Big)(t,x,z)\,\mathrm{d}z\,\mathrm{d}x \\ &\quad + \int \sigma_1(x-y)\sigma_2(z)\psi(t,y,z)|u(t,x)|^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &= \quad \mathcal{E}_{\mathrm{Schr}}(0).\end{aligned} \tag{14}$$

For the asymptotic Hartree equation (13), we get similarly

$$\mathcal{H}(t) = \frac{1}{2}\int |\nabla_x\tilde{u}(t,x)|^2\,\mathrm{d}x - \frac{\kappa}{2}\int \Sigma(x-y)|\tilde{u}(t,y)|^2|\tilde{u}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}y = \mathcal{H}(0). \tag{15}$$

Moreover, both quantum equations are invariant by translation and phase and conserve the mass of the wave function:

$$\mathscr{M}(t) = \int |u(t,x)|^2 \, \mathrm{d}x = \mathscr{M}(0), \qquad \tilde{\mathscr{M}}(t) = \int |\tilde{u}(t,x)|^2 \, \mathrm{d}x = \tilde{\mathscr{M}}(0). \tag{16}$$

However, there are fundamental differences between the two equations. Let

$$p(t) = \mathrm{Im} \int \nabla_x u(t,x) \bar{u}(t,x) \, \mathrm{d}x, \qquad \tilde{p}(t) = \mathrm{Im} \int \nabla_x \tilde{u}(t,x) \overline{\tilde{u}}(t,x) \, \mathrm{d}x$$

be the momentum associated to (1a)–(1b) and (13), respectively. We have, for (13),

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{p} = 0,$$

but

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = - \int_{\mathbb{R}^d} \nabla_x \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2(z) \psi(t,x,z) \, \mathrm{d}z \right) |u(t,x)|^2 \, \mathrm{d}x$$

for (1a)–(1b). We also introduce the center of mass

$$q(t) = \frac{\displaystyle\int_{\mathbb{R}^d} x \, |u(t,x)|^2 \, \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^d} |u(t,x)|^2 \, \mathrm{d}x} = \frac{1}{\mathscr{M}(0)} \int_{\mathbb{R}^d} x \, |u(t,x)|^2 \, \mathrm{d}x$$

associated to (1a)–(1b) and a similar definition $\tilde{q}(t)$ for (13). We have

$$\mathscr{M}(0) \frac{\mathrm{d}}{\mathrm{d}t} q(t) = p(t), \qquad \tilde{\mathscr{M}}(0) \frac{\mathrm{d}}{\mathrm{d}t} \tilde{q}(t) = \tilde{p}(t).$$

Therefore, the momentum conservation for (13) implies that the center of mass follows a straight line at constant speed. For (1a)–(1b), the analogy with the case of a single classical particle would lead to conjecture that the center of mass will stop exponentially fast. Numerical experiments shed some light on this issue [15]. Finally, we note that (13) is also Galilean invariant: if $\tilde{u}$ is a solution of (13), then $v(t,x) = \tilde{u}(t, x - tp_0) e^{ip_0 \cdot (x - t\frac{p_0}{2})}$ still is a solution of (13). This property is not fulfilled by the system (1a)–(1b), which leads to a specific behavior of the solutions, consistently with the previous remark.

1.2 Scaling properties

It is well-known that scaling invariance plays a central role in the analysis of non linear Schrödinger equations. Here, let (u, ψ) be a solution of (1a)–(1b), and, for given $\lambda, \mu > 0$, let us set

$$\big(u_{\lambda,\mu}(t,x), \psi_{\lambda,\mu}(t,x,z)\big) = \big(\mu \, u(\lambda^2 t, \lambda x), \mu \, \lambda^{n-1} \psi(\lambda^2 t, \lambda x, \lambda^2 z)\big).$$

It turns out that $u_{\lambda,\mu}$ is a solution of (1a)–(1b) but with the *rescaled* form functions

$$\sigma_1^{\lambda,\mu}(x) = \mu^{-1} \lambda^{d+1} \sigma_1(\lambda x) \quad \text{and} \quad \sigma_2^{\lambda}(z) = \lambda^{n+2} \sigma_2(\lambda^2 z).$$

Since σ_1 and σ_2 are not homogeneous functions, (u, ψ) and $(u_{\lambda,\mu}, \psi_{\lambda,\mu})$ are solutions of the *same* Schrödinger-Wave system if and only if $\lambda = 1 = \mu$. The same conclusion applies to the limiting system: if $\tilde{u}$ is a solution of (13) then $\tilde{u}_{\lambda,\mu}(t,x) = \mu\tilde{u}(\lambda^2 t, \lambda x)$ is a solution of (13) with the rescaled potential $\Sigma^{\lambda,\mu}(x) = \sigma_1^{\lambda,\mu} \star \sigma_1^{\lambda,\mu}(x) = \mu^{-2}\lambda^{d+2}\Sigma(\lambda x)$. Therefore, in contrast to the usual non linear Schrödinger or Hartree equations, we cannot find a relation between λ and μ such that the $(u_{\lambda,\mu}, \psi_{\lambda,\mu})$'s are solutions of the *same* equation than (u, ψ); this lack of scale invariance will have an important role in the sequel of this paper.

Nevertheless, the scaling property implies that any result valid for the Hartree equation with a given potential Σ equally applies to the equations with the modified potentials $\Sigma^{\lambda,\mu}$. Considering the case where $\lambda = \mu = \epsilon^{-1}$ and letting ϵ go to 0, up to a suitable renormalization, allows us to consider the regime $\Sigma \to \delta_0$ which formally leads to the standard cubic non linear Schrödinger equation

$$i\partial_t U + \frac{1}{2}\Delta_x U = -\kappa|U|^2 U. \tag{17}$$

This equation is L^2-sub-critical in the case $d = 1$, it is L^2-critical in the case $d = 2$ and L^2-super-critical in the case $d \geq 3$. Hence, this formal limit suggests different behaviors for the Hartree equation (when a smooth potential is considered), depending on the dimension d. Even if the continuity with respect to Σ as $\Sigma \to \delta_0$ is certainly wrong when $d \geq 2$ – (17) admits solutions which blow up in finite time while solutions of (13) are globally defined when Σ is smooth – our analysis shows several differences between the case $d = 1$ and $d \geq 2$, which can be understood from the formal asymptotic to (17). It is thus not surprising that our main results, Theorem 2.8 and Proposition 2.10, require some additional assumptions on the form function σ_1. Namely, in the case $d = 3$, we shall consider $\Sigma = \sigma_1 \star \sigma_1$ such that the rescaled potentials $\Sigma^{\lambda,\mu}$, with $\lambda, \mu > 0$, are *close enough* to $|\cdot|^{-1}$ (note that when $d = 3$ and $\Sigma = |\cdot|^{-1}$, the Hartree equation is L^2-sub-critical). When $d = 1$ we do not require any additional assumption on σ_1: see Proposition 2.15, obtained precisely by using the L^2 sub-critical feature of (17) when $d = 1$.

1.3 Solitary waves

The system (1a)–(1b) can be shown to be well-posed, in natural functional spaces associated to the energy conservation.

Theorem 1.1 *Let **(H1)**–**(H2)** be fulfilled. For all $u_0 \in H^1(\mathbb{R}^d_x)$, $\psi_0 \in L^2(\mathbb{R}^d_x; \dot{H}^1(\mathbb{R}^n_z))$ and $\psi_1 \in L^2(\mathbb{R}^d_x; L^2(\mathbb{R}^n_z))$, the system* (1a)–(1b) *and* (2) *admits a unique global solution (u, ψ) such that $u \in C^0([0, +\infty); H^1(\mathbb{R}^d_x))$ and*

$$\psi \in C^0\left([0, +\infty); L^2\left(\mathbb{R}^d_x; \dot{H}^1(\mathbb{R}^n_z)\right)\right) \cap C^1\left([0, +\infty); L^2\left(\mathbb{R}^d_x; L^2(\mathbb{R}^n_z)\right)\right).$$

The proof is detailed in Appendix A. The local well-posedness is based on Strichartz' estimates, which rely on the dispersive properties of the Schrödinger and the wave equations in the coupling. The difficulty comes from the fact that Strichartz' estimates for (1a) lead to estimates of u in $L^q_t L^r_x$ norms whereas Strichartz' estimates for (1b) lead to estimates on ψ in $L^r_x L^q_t L^p_z$ norms. Then, in order to gather these estimates, it is necessary to manage with permutations of Lebesgue-norms in time and space. For this purpose, assumption **(H2)** allows us to apply Hölder and Young inequalities in order to always obtain estimates in $L^q_t L^q_x$-norms. Eventually, that solutions are globally defined comes from the Hamiltonian structure of the system.

The main purpose of this article is to show the existence and the orbital stability of solitary waves for the Schrödinger-Wave system. Namely, we are going to study solutions of (1a)–(1b) with the form (3). The existence of such non dispersive solutions is the translation of the presence of some attractive dynamics induced by the model. The rescaling (7) is important in the discussion. We start by observing that if $(u, \psi) = (Q(x)e^{i\omega t}, \Psi(x, z))$ is a solution of (1a)–(1b), then (Q, Ψ) is a solution of

$$-\frac{1}{2}\Delta_x Q + \omega Q + \left(\sigma_1 \star_x \int \sigma_2 \Psi \,\mathrm{d}z\right) Q = 0, \qquad x \in \mathbb{R}^d \tag{18a}$$

$$-c^2 \Delta_z \Psi = -c^2 \sigma_2(z) \left(\sigma_1 \star_x Q^2\right)(x), \qquad x \in \mathbb{R}^d,\ z \in \mathbb{R}^n, \tag{18b}$$

which is in fact independent of the parameter c. In turn, the profiles (Q, Ψ) do not depend on c. Moreover these particular solutions $(Q(x)e^{i\omega t}, \Psi(x, z))$ are also solutions of the asymptotic system (12a)–(12b). It is therefore relevant to compare the behavior of the solutions of (1a)–(1b) and the solutions of (12a)–(12b) around the state $(Q(x)e^{i\omega t}, \Psi(x, z))$: this comparison provides information on the action of the environment on the quantum particle.

According to the previous discussion, the expected behavior for the Schrödinger wave system can be summarized as follows.

Conjecture 1.2 *Let* (Q, Ψ) *be a solution of* (18a)–(18b) *orbitally stable under the dynamic* (1a)–(1b)*. If* $u_0(x) = Q(x)e^{i\frac{p_0}{2}\cdot x}$ *for some sufficiently small* p_0 *and if* $(\psi_0, \psi_1) = (\Psi, 0)$*, then there exists two functions* $x = x(t)$ *and* $\gamma = \gamma(t)$ *such that*

- *the unique solution* (u, ψ) *of* (1a)–(1b) *associated to these initial conditions remains close (uniformly in time in some norms that have to be precised) to* $(Q(\cdot - x(t))e^{i\gamma(t)}, \Psi(\cdot - x(t), \cdot))$*;*
- $|\dot{x}(t)| \leq Ce^{-\lambda\frac{t}{c}}$ *and* $|x(t) - \bar{x}| \leq Ce^{-\lambda\frac{t}{c}}$*.*

Even if the orbital stability of solitary waves of non linear Schrödinger equations is a classical result for many years, see for instance [6, 40, 41], there are several difficulties to justify it in the present context. Firstly, we are dealing with a system and not with a mere scalar equation. Secondly, the nonlinearity is non local. Nevertheless, we can expect that structure properties of the simpler problem (13) still apply to the system (1a)–(1b). At first sight, assumption **(H2)** can be expected to make the problem easier than the case where Σ is replaced by the kernel of the Poisson equation in dimension $d = 3$, that is $\Sigma^0(x) = \frac{1}{|x|}$. This specific case (13) – the Schrödinger-Newton equation — has been investigated in detail by E. Lenzmann [19]. However, as reported above, while $\Sigma = \sigma_1 \star \sigma_1$ has better regularity and support properties, it does not satisfy any scale invariance. It turns out that the analysis of the Schrödinger-Newton equation exploits, in a quite crucial way, either explicit formula or the scale invariance which are very specific to the kernel $\frac{1}{|x|}$. For this reason, we shall use a quite indirect approach, that relies on the perturbative arguments developed in [19] for establishing spectral properties for the non relativistic Hartree equation. The second part of the conjecture justifies that the environment acts on the quantum particle as a friction force and will be the object of future investigations [15, 39].

2 Main results

As said above, the main objective is to discuss the existence and the stability of non trivial solutions (with finite mass and energy) of (1a)–(1b) with the form (3). In order to establish the existence, we start by observing that (Q, Ψ) has to be a solution of (18a)–(18b). Then we can express Ψ in term of Q as follows:

$$\Psi(x, z) = \Gamma(z)\, \sigma_1 \star Q^2(x),$$

where Γ stands for the unique solution of (9). Coming back to (18a), we deduce that Q satisfies

$$-\frac{1}{2}\Delta_x Q + \omega Q - \kappa(\Sigma \star Q^2)Q = 0 \tag{19}$$

with the definition (11). This equation is known as the *Choquard equation* and it has been intensively studied (see for example [27], [20] or [19] and the references therein). In particular, we already know from [27] that there exists infinitely many solitary waves.

2.1 Ground states

Nevertheless, we are only interested in *stable* solitary waves: for this reason, we consider solitary waves that minimize the energy of the system under a mass constraint, a quantity conserved by the evolution equation. Such solitary waves are called *ground states*. The specific case of the Newtonian potential $\Sigma^0(x) = \frac{1}{|x|}$ in dimension $d = 3$ has been studied in [20] which establishes the existence and uniqueness (up a change of phase and translation) of ground states for (13). The existence part of [20] still applies in the case where Σ is a smooth, compactly supported, radially symmetric, non increasing and non negative function. However, the arguments for proving the uniqueness part of the statement rely strongly on the specific form of the Newtonian potential. Besides, the definition of the energy functional for the system (1a)–(1b) differs from those of (13). Therefore, one has to check that (1a)–(1b) admits ground states. For that purpose we will need the following additional assumption on the form function σ_1.

(H3) The form function σ_1 is non increasing.

We interpret the energy functional (14) as depending on u, ψ and $\chi = \partial_t \psi$. Namely, for $u : \mathbb{R}^d \to \mathbb{C}$, $\psi, \chi : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$, we set

$$\begin{aligned} E(u, \psi, \chi) \quad = \quad & \frac{1}{2}\int |\nabla_x u(x)|^2 \,\mathrm{d}x + \frac{1}{2}\int \Big(\frac{|\chi|^2}{c^2} + |\nabla_z \psi|^2\Big)(x, z)\,\mathrm{d}z\,\mathrm{d}x \\ & + \int \sigma_1(x - y)\sigma_2(z)\psi(y, z)|u(x)|^2 \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

so that $\mathcal{E}_{\mathrm{Sch}}(t) = E(u, \psi, \partial_t\psi)(t)$. Similarly, we set

$$H(u) = \frac{1}{2}\int |\nabla_x u(x)|^2 \,\mathrm{d}x - \frac{\kappa}{2}\int \Sigma(x - y)|u(y)|^2|u(x)|^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{20}$$

see (15). In order to establish the existence of ground states we will study the following three minimization problems.

$$I_M := \inf \left\{ E(u,\psi,\chi) \ \text{ s.t. } \ (u,\psi,\chi) \in H^1_x \times L^2_x \dot{H}^1_z \times L^2_x L^2_z \text{ and } \|u\|^2_{L^2_x} \le M \right\}, \tag{21a}$$

$$J_M := \inf \left\{ E(u,\psi,\chi) \ \text{ s.t. } \ (u,\psi,\chi) \in H^1_x \times L^2_x \dot{H}^1_z \times L^2_x L^2_z \text{ and } \|u\|^2_{L^2_x} = M \right\}, \tag{21b}$$

$$K_M := \inf \left\{ E(u,\Gamma\,\sigma_1 \star |u|^2, 0) \ \text{ s.t. } \ u \in H^1_x \text{ and } \|u\|^2_{L^2_x} = M \right\}. \tag{21c}$$

The interest of (21c) comes from the fact that $E(u,\Gamma\,\sigma_1 \star |u|^2, 0) = H(u)$ since σ_1 is odd and therefore $\|\sigma_1 \star |u|^2\|^2_{L^2_x} = \iint |u|^2(x)\Sigma(x-y)|u|^2(y)\,\mathrm{d}x\,\mathrm{d}y$. Then, if K_M is reached at u, u is a ground state of (13) too and we will be able to compare ground states of (1a)–(1b) with ground states of (13). Section 3 is devoted to the proof of the following theorem.

Theorem 2.1 *Let **(H1)**–**(H3)** be fulfilled.*
(i) *For every $M \ge 0$, I_M is reached.*
(ii) *For every $M \ge 0$, $I_M = J_M = K_M$.*
(iii) *There exists a mass threshold $M_0 \ge 0$ such that for every $M \in [0, M_0]$, $J_M = 0$ and for every $M > M_0$, $J_M < 0$ is reached on $(u,\psi,\chi) = (u,\psi,0)$ with u non negative, radially symmetric and non increasing. Moreover (u,ψ) is a solution of* (18a)–(18b) *for a certain $\omega > 0$. In particular $\psi = \Gamma\,\sigma_1 \star |u|^2$ is non positive, u is an element of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and $K_M = J_M$ is reached at u.*
(iv) *If $d \ge 2$, then $M_0 > 0$.*

Note that we do not know whether the minimizer in item (iii) is uniquely defined, up to a possible change of phase and translation. Applying Lieb's method [20], we cannot even conclude whether or not the minimizer of J_M are radially symmetric, a preliminary step to establish uniqueness, and strictly positive. The alternative approach of L. Ma and L. Zhao [28, Section 5] provides a positive answer to the strict positivity and radial symmetry of the minimizer, though. Note also that the fourth item of this theorem is reminiscent to the fact that (1a)–(1b) does not have a scale invariance. We will see in the sequel that $M_0 = 0$ when $d = 1$, and this difference with the cases $d \ge 2$ can be related with the discussion of Section 1.2.

2.2 Orbital stability

The variational characterization will be used in Section 4 to establish the following orbital stability result for these ground states. In this statement, for a given mass $M > 0$, we denote by S_M the space of all possible ground states

$$S_M = \left\{ (\widetilde{Q},\widetilde{\Psi}) \in H^1_x \times L^2_x \dot{H}^1_z \ \text{ such that } \ \|\widetilde{Q}\|^2_{L^2_x} = M \text{ and } E(\widetilde{Q},\widetilde{\Psi},0) = J_M \right\}.$$

Theorem 2.2 *Let $M > M_0$ and (Q,Ψ) be in S_M. For every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $u_0 \in H^1_x$, $\psi_0 \in L^2_x \dot{H}^1_z$ and $\chi_0 \in L^2_x L^2_z$ with*

$$\|u_0 - Q\|^2_{H^1_x} + \|\psi_0 - \Psi\|^2_{L^2_x \dot{H}^1_z} + \|\chi_0\|^2_{L^2_x L^2_z} < \delta_\varepsilon,$$

then the unique solution $(u,\psi,\chi = \partial_t \psi)$ of (1a)–(1b) *with initial data (u_0,ψ_0,χ_0) satisfies*

$$\sup_{t \ge 0} \inf_{(\widetilde{Q},\widetilde{\Psi}) \in S_M} \left(\|u(t) - \widetilde{Q}\|^2_{H^1_x} + \|\psi(t) - \widetilde{\Psi}\|^2_{L^2_x \dot{H}^1_z} + \|\chi(t)\|^2_{L^2_x L^2_z} \right) < \varepsilon.$$

The proof is classical and based on the concentration-compactness lemma, see for instance [6, 23, 24] and the references therein. Since we do not know whether the ground states are unique (up to the equation invariants), the statement only tells us that a perturbation of a ground state stay close (uniformly in time) to *the manifold of all the possible ground states.* This is weaker than the expected conclusion which would assert that "a perturbation of a given ground state stay close (uniformly in time) to *the manifold generated by this ground state and the equation invariants (phase and translation)*".

2.3 Strengthened orbital stability

A strengthened result can be obtained by using an alternative approach, based on the study of the linearization of the energy around a ground state (see [30, 40, 41]; we also refer the reader to the lecture notes [29, Section 2.6] and the references therein). To be more specific, we fix $M > M_0$ and we consider a ground state (Q, Ψ) of J_M such that Q is positive, radially symmetric and decreasing and such that $\|Q\|^2_{L^2_x} = M$. We introduce

$$W(u, \psi, \chi) = E(u, \psi, \chi) + \omega \|u\|^2_{L^2_x}.$$

Next, we linearize this quantity around $(Q, \Psi, 0)$: for every $u \in H^1_x$, $\psi \in L^2_x \dot{H}^1_z$ and $\chi \in L^2_x L^2_z$, we have

$$\begin{aligned}
&W(Q+u, \Psi+\psi, \chi) = W(Q, \Psi, 0)\\
&+\frac{1}{2}\int_{\mathbb{R}^d} \nabla_x Q \cdot (\nabla_x u + \nabla_x \bar{u})\,\mathrm{d}x + \omega \int_{\mathbb{R}^d} Q(u+\bar{u})\,\mathrm{d}x + \int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2 \Psi\,\mathrm{d}z\right) Q(u+\bar{u})\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2 \psi\,\mathrm{d}z\right) Q^2\,\mathrm{d}x + \frac{1}{2}\iint_{\mathbb{R}^d \times \mathbb{R}^n} \nabla_z \Psi \cdot \nabla_z \psi\,\mathrm{d}x\,\mathrm{d}z\\
&+\frac{1}{2}\int_{\mathbb{R}^d} |\nabla_x u|^2\,\mathrm{d}x + \omega \int_{\mathbb{R}^d} |u|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2 \Psi\,\mathrm{d}z\right) |u|^2\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2 \psi\,\mathrm{d}z\right) Q(u+\bar{u})\,\mathrm{d}x + \frac{1}{2c^2}\iint_{\mathbb{R}^d \times \mathbb{R}^n} |\chi|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{1}{2}\iint_{\mathbb{R}^d \times \mathbb{R}^n} |\nabla_z \psi|^2\,\mathrm{d}x\,\mathrm{d}z\\
&+\int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2 \psi\,\mathrm{d}z\right) |u|^2\,\mathrm{d}x.
\end{aligned}$$

We write this as $W(Q+u, \Psi+\psi, \chi) = W(Q, \Psi, 0) + I_1 + ... + I_{12}$. Thanks to (18a), $I_1 + I_2 + I_3 = 0$ and thanks to (18b), $I_4 + I_5 = 0$. Let us denote

$$u = f + ig, \qquad f, g \in \mathbb{R}.$$

We can rewrite

$$I_6 + ... + I_{11} = \left\langle \mathcal{L}_+ \begin{pmatrix} f \\ \psi \end{pmatrix}, \begin{pmatrix} f \\ \psi \end{pmatrix} \right\rangle_{L^2_x \times L^2_x L^2_z} + \langle L_- g, g \rangle_{L^2_x} + \frac{1}{2c^2} \|\chi\|^2_{L^2_x L^2_z}$$

where

$$\mathcal{L}_+ = \begin{pmatrix} -\frac{1}{2}\Delta_x + \omega + \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2 \Psi\,\mathrm{d}z\right) & M_1 \\ M_2 & -\frac{1}{2}\Delta_z \end{pmatrix} \tag{22}$$

with

$$M_1\psi = \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\psi \,\mathrm{d}z\right) Q, \qquad M_2 f = \sigma_2\,(\sigma_1 \star Qf),$$

and

$$L_- = -\frac{1}{2}\Delta_x + \omega + \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\Psi \,\mathrm{d}z\right). \tag{23}$$

Let us also introduce the operator L_+ defined by

$$L_+ f = -\frac{1}{2}\Delta_x f + \omega f - \kappa(\Sigma \star Q^2)f - 2\kappa(\Sigma \star Qf)Q, \tag{24}$$

which will have an important role in the sequel: it is the analog to $\mathcal{L}_+$ for $\widetilde{W}(u) = H(u) + \omega\|u\|^2_{L^2_x}$. We eventually obtain the following decomposition

$$W(Q+u, \Psi+\psi, \chi) = W(Q,\Psi,0) + \left\langle \mathcal{L}_+ \begin{pmatrix} f \\ \psi \end{pmatrix}, \begin{pmatrix} f \\ \psi \end{pmatrix} \right\rangle_{L^2_x \times L^2_x L^2_z} + \langle L_- g, g\rangle_{L^2_x}$$
$$+ \frac{1}{2c^2}\|\chi\|^2_{L^2_x L^2_z} + \int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\psi \,\mathrm{d}z\right) |u|^2 \,\mathrm{d}x. \tag{25}$$

Remark 2.3 *Relation* (25) *holds true when replacing, for some $\alpha \in \mathbb{R}$, M_1 and M_2 in the definition of $\mathcal{L}_+$ by αM_1 and $(2-\alpha)M_2$. However, $\mathcal{L}_+$ is self-adjoint only in the particular case $\alpha = 1$.*

The key argument to prove an orbital stability result is to characterize the kernel of L_- and $\mathcal{L}_+$ and to prove that these operators are coercive under some orthogonality conditions. The operator L_- is a local operator, and we already have at hand the following statement, see for example [40].

Lemma 2.4 *We have* $\mathrm{Ker}(L_-) = \mathrm{Span}\{Q\}$ *and there exists a universal constant $\mu > 0$ such that for every $g \in H^1_x$,*

$$\langle L_- g, g\rangle_{L^2_x} \geq \mu\|g\|^2_{H^1_x} - \frac{1}{\mu}\left|\langle g, Q\rangle_{H^1_x}\right|^2. \tag{26}$$

The difficult part is to obtain an analogous statement for $\mathcal{L}_+$. The method consists in working on the operator L_+: the knowledge of the kernel of L_+ will allow us to identify the kernel of $\mathcal{L}_+$ and a coercivity property for L_+ will provide a coercivity property for $\mathcal{L}_+$ too. By direct inspection, it can be checked that $\mathrm{Span}\{\partial_{x_j}Q\,,\, j=1,\ldots,d\} \subset \mathrm{Ker}(L_+)$; we shall work further to establish the reverse inclusion and characterize $\mathrm{Ker}(L_+)$. Since L_+ is a non-local operator, classical arguments based on Sturm-Liouville theory are not applicable. We shall need to develop alternative approaches and perturbative arguments, inspired form [19].

We are going to exploit results known for some limiting cases, depending on the dimension d. Namely, in the case $d = 1$ we will consider the case of the delta function

$$\Sigma^0 = \delta_0, \tag{27}$$

while in dimension $d = 3$ we will consider the case of the Newtonian potential

$$\Sigma^0(x) = \frac{1}{|x|}. \tag{28}$$

Indeed, for these specific situations the following statement holds.

Lemma 2.5 *Let $d=1$ with the potential* (27) *or $d=3$ with the potential* (28). *We have* $\mathrm{Ker}(L_+) = \mathrm{Span}\{\partial_{x_j} Q\,,\, j=1,\ldots,d\}$. *Moreover, there exists a universal constant $\nu>0$ such that for every $f \in H^1_x$,*

$$\langle L_+ f, f\rangle_{L^2_x} \geq \nu \|f\|^2_{H^1_x} - \frac{1}{\nu}\left(\left|\langle f, Q\rangle_{L^2_x}\right|^2 + \sum_{j=1}^{d} \left|\langle f, \partial_{x_j} Q\rangle_{L^2_x}\right|^2 \right). \tag{29}$$

In the case $d=1$, the result is well known since the paper of M. Weinstein [40]. The analysis of the case $d=3$ is quite recent: the characterization of the kernel of L_+ has been obtained by E. Lenzmann in [19] and then, based on this characterization, P. D'Avenia and M. Squassina [7] established the coercivity property (29). We need to extend such a property to potentials with the form $\Sigma = \sigma_1 \star \sigma_1$: we denote by $\mathscr{A}_d$ the set of *admissible* form functions σ_1 such that Lemma 2.5 applies in dimension d when $\Sigma = \sigma_1 \star \sigma_1$. This is made clear by the following Definition.

Definition 2.6 *We say that σ_1 is an admissible form function if it satisfies **(H2)**–**(H3)** and if there exists a mass interval I of non empty interior such that for every $M \in I$ and every positive and radially symmetric minimizer Q_M of K_M, Lemma 2.5 applies.*

That $\mathscr{A}_d$ is non empty is highly non trivial: in [19] the characterization in Lemma 2.5 relies strongly on the specific form of the Newtonian potential and the scale invariance property of equation (19) in this specific case. We let this question open for a while and state the following lemma which links the properties of the operator $\mathcal{L}_+$ to the properties of the operator L_+. Note that from now on we denote

$$\mathscr{H} = \{(u,\psi) \in H^1_x \times L^2_x \dot{H}^1_z\}$$

which is a Hilbert space when endowed with the norm defined by

$$\|(u,\psi)\|^2_{\mathscr{H}} = \|u\|^2_{H^1_x} + \|\psi\|^2_{L^2_x \dot{H}^1_z}.$$

Lemma 2.7 *Assume **(H1)**–**(H3)**. Let $\sigma_1 \in \mathscr{A}_d$ be an admissible form function and assume that the mass M of the considered ground state Q is in the interval I of Definition 2.6. Then* $\mathrm{Ker}(\mathcal{L}_+) = \mathrm{Span}\{(\partial_{x_j} Q, \partial_{x_j}\Psi)^t\,,\, j=1,\ldots,d\}$ *and there exists a universal constant $\tilde{\nu}>0$ such that for every $(f,\psi) \in \mathscr{H}$,*

$$\left\langle \mathcal{L}_+ \begin{pmatrix} f \\ \psi \end{pmatrix}, \begin{pmatrix} f \\ \psi \end{pmatrix} \right\rangle_{L^2_x \times L^2_x L^2_z} \geq \tilde{\nu} \|f,\psi\|^2_{\mathscr{H}} - \frac{1}{\tilde{\nu}}\left(\left|\langle f, Q\rangle_{L^2_x}\right|^2 + \sum_{j=1}^{d} \left|\langle f, \partial_{x_j} Q\rangle_{L^2_x}\right|^2 \right). \tag{30}$$

This lemma is the key ingredient to prove the following orbital stability theorem that strengthens Theorem 2.2. The proof is detailed in Section 5.

Theorem 2.8 *Assume **(H1)**–**(H3)**. Let $\sigma_1 \in \mathscr{A}_d$ be an admissible form function and assume that $\|Q\|^2_{L^2_x} \in I$. For every $(u_0,\psi_0,\chi_0) \in H^1_x \times L^2_x \dot{H}^1_z \times L^2_x L^2_z$ let us denote by $(u,\psi,\chi=\partial_t\psi)$ the unique solution of* (1a) *and* (1b) *associated to the initial data (u_0,ψ_0,χ_0). Let us assume $\|u_0\|_{L^2_x} = \|Q\|_{L^2_x}$. There exists $\varepsilon_0>0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ we can find $\eta(\varepsilon)>0$ and $\delta(\varepsilon)>0$ such that, if*

$$\|u_0 - Q, \psi_0 - \Psi\|^2_{\mathscr{H}} + \frac{1}{c^2}\|\chi_0\|^2_{L^2_x L^2_z} \leq \eta(\varepsilon)^2 \quad \textit{and} \quad W(u_0,\psi_0,\chi_0) - W(Q,\Psi,0) \leq \delta(\varepsilon),$$

then there exists two functions $x(t)$ and $\gamma(t)$, continuous in time, such that for every $t \geq 0$, $v = e^{-i\gamma(t)}u(t, \cdot + x(t))$ satisfies the following orthogonality conditions

$$\left\langle \operatorname{Re} v, \partial_{x_j} Q \right\rangle_{L^2_x} = 0, \qquad j = 1, \ldots, d, \tag{31a}$$

$$\langle \operatorname{Im} v, Q \rangle_{H^1_x} = 0 \tag{31b}$$

and

$$\sup_{t \geq 0} \left\| u(t) - e^{i\gamma(t)}Q(\cdot - x(t)), \psi(t) - \Psi(\cdot - x(t)) \right\|^2_{\mathscr{H}} + \frac{1}{c^2}\|\chi(t)\|^2_{L^2_x L^2_z} \leq \varepsilon^2.$$

Remark 2.9 *Note that in the regime $c \gg 1/\varepsilon^2$, the theorem still applies if the perturbation χ_0 is not close to zero. It is also worth remarking that $\eta(\varepsilon)$ and $\delta(\varepsilon)$ are uniform with respect to c.*

It is worth commenting the assumption on the mass of u_0 which did not appear in Theorem 2.2. Usually, assuming $\|u_0\|_{L^2_x} = \|Q\|_{L^2_x}$ is not a restriction. Indeed, as soon as the definition of the map $M \mapsto Q_M$ is meaningful (*i.e.* when ground states are unique or at least locally unique) and defines a continuous map, any small perturbation u_0 of a ground state Q_M is also a small perturbation of the ground state $Q_{\|u_0\|_{L^2_x}}$. Here, relaxing this assumption requires to justify, first, that the ground states are (at least locally) unique and, second, their continuity with respect to the mass M. We decided not to focus on the uniqueness issues in this work; nevertheless we can provide some hints. Our approach to find admissible form functions σ_1 is inspired by the strategy developed by E. Lenzmann [19] in order to prove the uniqueness of ground states (for almost every sufficiently small mass M) for the non relativistic Hartree equation. Therefore, it is likely that a similar result applies here for almost every $M \in I$. Working in this direction may probably allow us to justify that the assumption on the mass of the perturbation u_0 is indeed not a restriction.

Theorem 2.8 becomes fully meaningful if we are able to characterize the set of admissible form function $\mathscr{A}_d$, or at least to justify that $\mathscr{A}_d$ contains physically relevant form functions σ_1. This is the purpose of the following sections which contain the most original insights of the paper. Our results cover the three-dimensional case $d = 3$, which is the most relevant physically, and the one-dimensional case $d = 1$ for which numerical investigations is more affordable [15]. It is worthwhile to see the role of the space dimension in the analysis, and we also provide some hints on the case $d = 2$.

2.4 The case $d = 3$

Section 8 is devoted to the construction of admissible form functions σ_1 in dimension $d = 3$. The difficulty in identifying the class of admissible form functions σ_1 is a weakness of the method compared to the approach by concentration-compactness. Nevertheless this additionnal restriction allows us to obtain the more precise orbital stability result of Theorem 2.8 and we shall see in Section 8 that we can find many form functions σ_1 that fit the physical framework introduced in [3]. We proceed in two steps. The idea is to boil down a perturbative approach for potentials Σ close, in an appropriate sense, to $|\cdot|^{-1}$, and then to push this result by suitable rescalings which allow us to identify physically relevant potentials $\Sigma = \sigma_1 \star \sigma_1$ not necessarily close to $|\cdot|^{-1}$. An important issue in this approach is to clarify the role of the mass constraint: Theorem 2.2 applies to any ground state of mass $M > M_0$. Hence, we expect stability results that apply to a continuum of possible masses M, as stated in Definition 2.6.

Proposition 2.10 *The set $\mathscr{A}_3$ of admissible form functions is non empty.*

We will indeed see in Section 8 that the set $\mathscr{A}_3$ contains at least every form function σ_1 satisfying **(H2)**-**(H3)** and such that $\Sigma^{\lambda,\mu}(x) = \mu\Sigma(\lambda x)$ is *close enough* to $\Sigma^0 = |\cdot|^{-1}$ for suitable rescaling parameters $\lambda, \mu > 0$. As explained above, our strategy to identify admissible form functions and to establish the orbital stability for the Schrödinger-Wave system is based on a perturbative analysis from Σ^0. For this purpose let us introduce the following more precise notations.

Definition 2.11 *For a given potential Σ we denote H^Σ and K_M^Σ the corresponding energy defined by* (20)*, and the minimization problem* (21c)*, respectively. Then we denote by Q_M^Σ a positive and radially symmetric minimizer of K_M^Σ and by $\omega(\Sigma, Q_M^\Sigma)$ the constant $\omega > 0$ such that Q_M^Σ is a solution of* (19) *with Σ and $\omega = \omega(\Sigma, Q_M^\Sigma)$. Note that the notation Q_M^Σ could design several minimizers since* a priori *we do not get the uniqueness of the minimizers of K_M^Σ. Moreover we make precise how the operator L_+ defined by* (22) *depends on Σ, Q and ω. Since we will only consider cases where $\omega = \omega(\Sigma, Q)$ we will use the notation $L_+ = L_+(\Sigma, Q)$.*

We consider a sequence $(\Sigma^\varepsilon)_{\varepsilon>0}$ of smooth potentials satisfying the following assumption:

(H4) For every ε there exists σ_1^ε satisfying **(H2)**–**(H3)** such that $\Sigma^\varepsilon = \sigma_1^\varepsilon \star \sigma_1^\varepsilon$ and the sequence $(\Sigma^\varepsilon)_{\varepsilon>0}$ converges to $\Sigma^0 = |\cdot|^{-1}$ in the following sense: for every $R > 0$,

$$\|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|\le R}\|_{L_x^{3/2}} + \|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|>R}\|_{L_x^\infty} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{32}$$

For such family we know that for each $\varepsilon > 0$, there exists a mass threshold $M_0^\varepsilon > 0$ such that $K_M^{\Sigma^\varepsilon}$ is achieved for every $M > M_0^\varepsilon$. In order to work with a fixed mass $M > 0$ we will also assume that $\sup_{0<\varepsilon\le 1}(M_0^\varepsilon) < +\infty$ and we will consider a mass M such that $M > \sup(M_0^\varepsilon)$. This assumption is quite reasonable since $\Sigma^\varepsilon \to \Sigma^0$ and there is no mass threshold in the case $\Sigma = \Sigma^0$. We refer the reader to Lemma 7.1 which insures that this assumption is indeed always valid in the previous context.

Then we consider a sequence $(Q^\varepsilon)_{\varepsilon>0}$ of smooth, positive, radially symmetric and decreasing functions and a sequence $(\omega^\varepsilon)_{\varepsilon>0}$ of positive numbers such that $Q^\varepsilon = Q_M^{\Sigma^\varepsilon}$ and $\omega^\varepsilon = \omega(\Sigma^\varepsilon, Q_M^{\Sigma^\varepsilon})$. In particular each Q^ε is a solution of (19) with $\Sigma = \Sigma^\varepsilon$ and $\omega = \omega^\varepsilon$. We also consider Q^0, the unique positive and radially symmetric minimizer of $K_M^{\Sigma^0}$. Note that Q^0 is also decreasing and we can find $\omega^0 > 0$ such that Q^0 is a solution of (19) with $\Sigma = \Sigma^0$ and $\omega = \omega^0$. Hence, the cornerstone of the analysis is given by the following result, established in Section 7.

Proposition 2.12 *With the previous notations and assuming moreover **(H4)**, the following properties hold.*
(i) Convergence. *For every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$,*

$$\|Q^\varepsilon - Q^0\|_{H_x^1} + |\omega^\varepsilon - \omega^0| < \delta.$$

(ii) Coercivity. *There exists $\bar\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \bar\varepsilon_0)$, $Q^\varepsilon = Q_M^{\Sigma^\varepsilon}$ and $\omega^\varepsilon = \omega(\Sigma^\varepsilon, Q_M^{\Sigma^\varepsilon})$ there exists $\nu(\Sigma^\varepsilon, Q^\varepsilon, \omega^\varepsilon) > 0$ satisflying, for every $f \in H_x^1$,*

$$\langle L_+(\Sigma^\varepsilon, Q^\varepsilon, \omega^\varepsilon)f, f\rangle_{L_x^2} \ge \nu(\Sigma^\varepsilon, Q^\varepsilon, \omega^\varepsilon)\|f\|_{H_x^1}^2 - \frac{1}{\nu^0}\left(\left|\langle f, Q^\varepsilon\rangle_{L_x^2}\right|^2 + \sum_{j=1}^3 \left|\langle f, \partial_{x_j} Q^\varepsilon\rangle_{L_x^2}\right|^2\right),$$

where ν^0 is the best constant possible in Lemma 2.5. Moreover, $\nu(\Sigma^\varepsilon, Q^\varepsilon, \omega^\varepsilon) \nearrow \nu^0$ when $\varepsilon \to 0$. This coercivity inequality insures that the kernel of $L_+(\Sigma^\varepsilon, Q^\varepsilon, \omega^\varepsilon)$ is spanned by the $\partial_{x_j} Q^\varepsilon$ and Lemma 2.5 applies to the kernel Σ^ε as well.

Remark 2.13 *In point (i), ε_0 depends on the chosen sequence $(Q^\varepsilon)_{\varepsilon>0}$ whereas in point (ii), $\bar{\varepsilon}_0$ is the same for every sequence $(Q^\varepsilon)_{\varepsilon>0}$. However, how the coercivity constant $\nu(\Sigma^\varepsilon, Q^\varepsilon, \omega^\varepsilon)$ converges to ν^0 depends on the considered sequence.*

In this proposition, how small $\bar{\varepsilon}_0$ has to be depends on M; hence the result cannot be extended to consider, for a fixed potential Σ^ε close to Σ^0, a continuum of possible masses M. The statement applies for a given mass M but it is not sufficient to justify that $\mathscr{A}_3$ is non empty. This issue is addressed in Section 8.

Remark 2.14 *Our approach can be adapted to treat many problems involving a non local definition of the potential, without scale invariance. A relevant example is the case of the Hartree equation with the Yukawa potential $\Sigma(x) = \frac{e^{-\mu|x|}}{|x|}$, which corresponds to a coupling between the Schrödinger equation and the screened Poisson equation $\mu^2\Phi - \Delta_x\Phi = |u|^2$ for the potential. The stability analysis for this problem is performed by a variational approach in [43] and an improved statement has been obtained in [17] by using a perturbative approach next to $\mu = 0$.*

2.5 The case $d = 1$

The characterization of $\mathscr{A}_1$ is much more easier. This is related to the remarks made in Section 1.2. Indeed, we can adapt the same strategy than developed for $d = 3$, but now considering perturbations around δ_0, and using the fact that the cubic non linear Schrödinger equation is L^2-sub-critical for $d = 1$. We obtain the following result.

Proposition 2.15 *If σ_1 satisfies **(H2)-(H3)**, then $\sigma_1 \in \mathscr{A}_1$. Moreover there exists a mass $M^* > 0$ such that $(0, M^*) \subset I$. As a consequence, we obtain $M_0 = 0$.*

Let σ_1 satisfy **(H2)-(H3)** and consider the sequence $(\Sigma^\varepsilon)_{\varepsilon>0}$ of smooth potentials defined by

$$\Sigma^\varepsilon(x) = \varepsilon^{-1}\Sigma(\varepsilon^{-1}x), \quad \Sigma = \sigma_1 \star \sigma_1.$$

This sequence converges to δ_0. We know that for each $\varepsilon > 0$, there exists a mass threshold $M_0^\varepsilon \geq 0$ such that $K_M^{\Sigma^\varepsilon}$ is achieved for every $M > M_0^\varepsilon$. As in the case $d = 3$, we can prove (thanks to an easy adaptation of Lemma 7.1) that $\sup_\varepsilon(M_0^\varepsilon) < +\infty$.

Remark 2.16 *Thanks to the scaling relations of Section 1.2, applied with $\mu = \lambda = \varepsilon^{-1}$, we can express M_0^ε in terms of ε and M_0^1: $M_0^\varepsilon = \varepsilon^{-1}M_0^1$. Combining this relation to the boundedness of $\sup_\varepsilon(M_0^\varepsilon)$ implies $M_0^1 = 0$ and then $M_0^\varepsilon = 0$.*

Hence, for a given mass $M > 0$ we can consider a sequence of ground states $Q^\varepsilon = Q_M^{\Sigma^\varepsilon}$ and Lagrange multipliers $\omega^\varepsilon = \omega(\Sigma^\varepsilon, Q_M^{\Sigma^\varepsilon})$ and we can justify that the conclusions of Proposition 2.12 (where Q^0 is now the unique positive and even minimizer of $K_M^{\delta_0}$ and ω^0 is the corresponding Lagrange multiplier) also hold in this case. We refer the reader to Example 3 in Section 8 where we briefly justify how the conclusions of Proposition 2.12 allow us to obtain Proposition 2.15.

Remark 2.17 *There is no major difficulty in order to adapt the proof of Proposition 2.12 to the case $d = 1$. The compact embedding $H^1_{rad}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $p \in (2, p_c)$ holds in dimension $d \geq 2$, but we can exploit the fact that each Q^ε is decreasing in order to recover some compactness result (as in the proof of Theorem 2.1-(i)).*

2.6 The case $d = 2$

One may naturally wonder what happens in dimension $d = 2$. The discussion in Section 1.2 supports the intuition that the case $d = 2$ is likely more intricate than $d = 1$ and studying this situation can shed some light on the restrictions on σ_1 adopted when $d = 3$ (compare the characterization of the set of admissible form functions for $d = 1$ in Proposition 2.15 to the weaker statement in Proposition 2.10).

The role of the dimension d appears in the analysis of the minimisation problem for K_M: for $d = 1$, the mass threshold M_0 is zero, while it is strictly positive in higher dimensions. We can thus expect to obtain useful information by studying more precisely the value of M_0 and considering ground states having a mass close to M_0. Going back to the proof of $M_0 > 0$ (see the proof of Lemma 3.1-f)), we are led to study the best constant $C > 0$ in the inequality

$$\left| \int (\Sigma \star |u|^2)\, |u|^2 \,\mathrm{d}x \right| \leq C \|\nabla_x u\|^2_{L^2_x} \|u\|^2_{L^2_x}.$$

This yields to the minimization problem

$$a^\Sigma := \inf_{\substack{u \in H^1_x \\ u \neq 0}} A^\Sigma(u), \qquad A^\Sigma(u) = \frac{\|\nabla_x u\|^2_{L^2_x} \|u\|^2_{L^2_x}}{\displaystyle\int (\Sigma \star |u|^2)\, |u|^2 \,\mathrm{d}x}.$$

Indeed, we can prove that $M_0 = a^\Sigma/\kappa$ and we are thus led to *compute* a^Σ. Coming back to the scaling relations discussed in Section 1.2, we set $\Sigma^\varepsilon(x) = \varepsilon^{-2}\Sigma(\varepsilon^{-1}x)$, $u^\varepsilon(x) = \varepsilon^{-1}u(\varepsilon^{-1}x)$ and we check that $A^{\Sigma^\varepsilon}(u^\varepsilon) = A^\Sigma(u)$. Accordingly, we have $a^{\Sigma^\varepsilon} = a^\Sigma$ for every $\varepsilon > 0$. Passing *formally* to the limit $\varepsilon \to 0$ in this relation, which amounts to saying $\Sigma^\varepsilon \to \delta_0$ (note that up to change the value of κ, we can always assume that $\|\Sigma\|_{L^1_x} = 1$), would lead to identify a^Σ and a^{δ_0} where a^{δ_0} stands for the best constant in the Gagliardo-Nirenberg inequality

$$\|u\|^4_{L^4_x} \leq \frac{1}{a^{\delta_0}} \|\nabla_x u\|^2_{L^2_x} \|u\|^2_{L^2_x}, \qquad a^{\delta_0} = \inf_{\substack{u \in H^1_x \\ u \neq 0}} A^{\delta_0}(u).$$

It is well known that A^{δ_0} admits minimizers, see the pioneering work [42] which points out the connection to the non linear Schrödinger equation, and the recent reviews [2, 10] ; these minimizers are of arbitrary mass (thanks to the relation $A^\Sigma(\theta u) = A^\Sigma(u)$ for every $\theta \neq 0$) and solution of the equation

$$-\frac{1}{2}\Delta_x Q + \frac{1}{2} \frac{\|\nabla_x Q\|^2_{L^2_x}}{\|Q\|^2_{L^2_x}} Q - \frac{a^{\delta_0}}{\|Q\|^2_{L^2_x}} Q^3 = 0.$$

By considering a minimizer of mass a^{δ_0} and thanks to the rescaling $Q^\lambda(x) = \lambda Q(\lambda x)$ (which leaves both the equation and the mass of the minimizer invariant) we can simply consider the equation

$$-\frac{1}{2}\Delta_x Q + Q - Q^3 = 0.$$

It is well known that this equation admits a unique positive and radially symmetric solution, see [18, 31]. By denoting Q^{δ_0} this unique solution we eventually obtain $M_0 = \|Q^{\delta_0}\|_{L^2_x}^2/\kappa$. This discussion makes formally a bridge appear with the asymptotic system (17) when $\Sigma \to \delta_0$; this intuition might be a guide for further analysis, which relies on the following known results for the cubic non linear Schrödinger equation in dimension $d = 2$, where we keep the notations of Definition 2.11. Details can be found in the seminal paper [32], and in the in-depth review [34] which contains complete references.

Theorem 2.18 *Let $d = 2$ and $M_0 = \|Q^{\delta_0}\|_{L^2_x}^2/\kappa$. The following assertion hold:*

i) For every $0 \le M \le M_0$, $K_M^{\delta_0} = 0$ while $K_M^{\delta_0} = -\infty$ when $M > M_0$.

ii) If $u \in H^1_x$ is such that $0 < \|u\|_{L^2_x}^2 \le M_0$ and $H^{\delta_0}(u) = 0$, then there exists $\lambda_0 > 0$, $x_0 \in \mathbb{R}^2$ and $\gamma_0 \in \mathbb{R}$ such that

$$u(x) = \frac{\lambda_0}{\sqrt{\kappa}}\, Q^{\delta_0}(\lambda_0 x - x_0)e^{i\gamma_0}.$$

As a consequence $\|u\|_{L^2_x}^2 = M_0$.

iii) Let $L_+^{\delta_0} := L_+(\delta_0, Q^{\delta_0}/\sqrt{\kappa})$. There exists a universal constant $\nu > 0$ such that for every $f \in H^1_x$,

$$\left\langle L_+^{\delta_0} f, f\right\rangle_{L^2_x} \ge \nu\|f\|_{H^1_x}^2 - \frac{1}{\nu}\left(\left|\langle f, Q^{\delta_0}\rangle_{L^2_x}\right|^2 + \sum_{j=1}^{2}\left|\langle f, \partial_{x_j} Q^{\delta_0}\rangle_{L^2_x}\right|^2 + \left|\langle f, x\cdot\nabla_x Q^{\delta_0} + Q^{\delta_0}\rangle_{L^2_x}\right|^2\right). \tag{33}$$

iv) If $\|u_0\|_{L^2_x}^2 < M_0$, then the unique solution u of (17) with initial data u_0 satisfies the following scattering estimate: there exists $u_\infty \in H^1_x$ such that

$$\|u(t) - S(t)u_\infty\|_{H^1_x} \underset{t\to+\infty}{\longrightarrow} 0,$$

where $S(t)u_\infty$ stands for the unique solution of the linear Schrödinger equation with initial data u_∞.

v) If $\|u_0\|_{L^2_x}^2 = M_0$, then there are only three possible scenario for the unique solution u of (17) associated to the initial data u_0:

- *u is a solitary wave (up to the equation's invariants),*
- *u blows up in finite time,*
- *u is globally defined in time and satisfies the scattering property.*

Here, we have obtained the analogue of point (i) when a smooth potential Σ replaces the delta function δ_0: in this case the only difference is that K_M^Σ is finite and strictly negative when $M > M_0$.

Point (ii) gives the characterization of the manifold of all possible ground states of mass M_0. Compared to the case $d = 1$, here the manifold is parametrized by an additional parameter ($\lambda_0 \in \mathbb{R}_+^*$) which is the translation of the L^2-criticality of this case. Hence the coercivity relation in (iii) naturally involves an additional orthogonality condition, compared to (29). From this discussion, we can address the following questions for future investigations.

Question 1. Does A^Σ admit minimizers ? If it is so, what is the dimension of the manifold of all minimizers ? It involves at least three free parameters (one for the phase and two for the translation), but does it need an additional parameter λ_0 ? In other words, does it exist a transformation $\mathcal{T}_{\lambda_0}$, continuous with respect to the one-dimensional parameter λ_0, which does not correspond to a translation or a change of phase, and such that if Q is a ground state then $\mathcal{T}_{\lambda_0}Q$ is a ground state too ? Moreover, if such a transformation exists, does it conserve the L^2-norm of its argument (for every λ_0, $\|\mathcal{T}_{\lambda_0}Q\|_{L^2_x} = \|Q\|_{L^2_x}$) ?

Depending on the answers, it will be possible to obtain a coercivity relation of the form (29) or with an additional orthogonality relation as in (33). Then, such results will allow us to justify the (un-)stability of ground states of mass M_0. Note that the existence of a transformation $\mathcal{T}_{\lambda_0}$ is quite natural. Indeed, the continuity of the ground states with respect to their mass is at least expected in any dimension d. The main issue is to determine whether or not the transformation also preserves the mass, as it does for the formal limit case δ_0.

Question 2. Is it possible to extend the conclusions for ground states of mass M_0 (if they do exist) to ground states of mass $M > M_0$ close to M_0 ?

Question 3. Does the analogue of point (iv) still hold true when u is a solution of (13) ?

Thanks to the smoothness of the potential Σ, we already know that every solution of (13) is globally defined in time. This excludes the scenario where solutions blow up in finite time. This is a major difference between the dynamics of (17) and (13). Nevertheless, the similar structure of the infimum of their energy when $M < M_0$ suggests that solutions of (13) with a mass strictly less than M_0 obey the scattering property. If it is so, this major difference with the case $d = 1$ (for which ground states exist for any mass) would indicate that dynamics specific to L^2-critical or L^2-super-critical equations also hold when $d \geq 2$ and (13) is considered with a smooth potential Σ. As a consequence, obtaining positive results of stability when $d \geq 2$ seems much more challenging than in the case $d = 1$.

In this paper this difficulty is treated at the price of restricting to potentials Σ *close* to $|\cdot|^{-1}$ (instead of being close to δ_0). This viewpoint takes advantage of the fact that (13) is L^2-sub-critical when $d = 3$ and $\Sigma = |\cdot|^{-1}$, which allows us to proceed with a pertubative analysis. Equation (13) is equally L^2-sub-critical when $\Sigma = |\cdot|^{-\alpha}$ with $1 \leq \alpha < 2$ (the case $\alpha = 2$ being L^2-critical): up to the knowledge of stability results for these cases, the strategy developed in the paper could be adapted to any potential Σ close to $|\cdot|^{-\alpha}$, when $1 \leq \alpha < 2$.

3 Existence of ground states: proof of Theorem 2.1

Let us gather the basic properties of I_M, J_M and K_M in the following lemma, which is further illustrated by Fig. 1.

Lemma 3.1 *Let **(H1)**–**(H2)** be fulfilled. The following assertions hold:*
a) *$M \mapsto I_M$ is non increasing.*
b) *$I_0 = J_0 = 0$ are reached at $(u, \psi, \chi) = (0, 0, 0)$ and $K_0 = 0$ is reached at $u = 0$.*
c) *For every $M \geq 0$, $-\infty < I_M \leq J_M \leq K_M \leq 0$.*
d) *There exists a mass threshold $M_0 \geq 0$ such that $I_M = 0$ for $M \in [0, M_0]$ and $I_M < 0$ for $M > M_0$.*
e) *If $I_M < 0$ is reached at (u, ψ, χ), then $\|u\|_{L_x^2}^2 = M$ and $J_M = I_M$ is reached at (u, ψ, χ). Moreover $\chi = 0$, $\psi = \Gamma\, \sigma_1 \star |u|^2$ and $u \in \mathcal{S}(\mathbb{R}^d)$ is a solution of* (19) *for a certain $\omega > 0$. In particular $K_M = J_M$ is reached at u.*
f) *If $d \geq 2$, then $M_0 > 0$.*

Before proving this lemma let us make several remarks

- Points c) and e) coupled with Theorem 2.1-(i) imply $K_M = J_M = I_M$ for every $M \geq 0$.
- Points d) and e) coupled with Theorem 2.1-(i) imply that J_M is reached for $M > M_0$ and improve also point a): $I_M = 0$ for $M \in [0, M_0]$ and $M \mapsto I_M$ is strictly decreasing on $(M_0, +\infty)$.
- The proof of point f) will give us the following additional information on M_0:

$$0 < \frac{1}{\kappa C^2 \|\Sigma\|_{L_x^{\frac{d}{2}}}} \leq M_0. \tag{34}$$

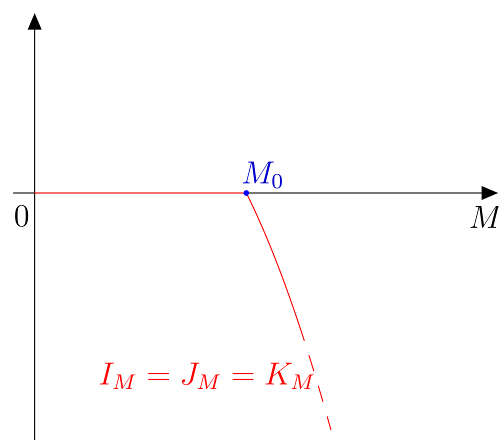

Figure 1: A possible graph representing I_M, J_M, K_M as a function of the mass M. Note that nothing ensures that these functions are differentiable as the picture might indicate.

Proof. *Items a) and b)* are direct consequences of the definition of I_M, J_M and K_M. The non trivial parts of c) are to prove that $E(u, \psi, \chi)$ is bounded from below under the mass constrain

$\|u\|^2_{L^2_x} = M$ and that $K_M \le 0$. Since for every (u,ψ,χ),

$$\begin{aligned} E(u,\psi,\chi) &\ge \frac12\|\nabla_x u\|^2_{L^2_x} - \left|\int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\psi\,\mathrm{d}z\right)|u|^2\,\mathrm{d}x\right| + \frac12\|\nabla_z\psi\|^2_{L^2_xL^2_z} + \frac{1}{2c^2}\|\chi\|^2_{L^2_xL^2_z} \\ &\ge \frac12\|\nabla_x u\|^2_{L^2_x} - M\,\|\sigma_1\|_{L^2_x}\|\sigma_2\|_{L_z^{2n/(n+2)}}\|\psi\|_{L^2_xL_z^{2n/(n-2)}} + \frac12\|\nabla_z\psi\|^2_{L^2_xL^2_z} + \frac{1}{2c^2}\|\chi\|^2_{L^2_xL^2_z}, \end{aligned} \tag{35}$$

the Sobolev inequality $\|f\|_{L_z^{2n/(n-2)}} \lesssim \|\nabla_z f\|_{L^2_z}$, see e.g. [33, Theorem, p. 125] allows us to conclude that $I_M > -\infty$. In order to prove $K_M \ge 0$ we use the immediate estimate $H(u) \le \|\nabla_x u\|^2_{L^2_x}/2$. Then, for every $u \in H^1_x$, by setting $u_\lambda(x) = \lambda^{d/2}u(\lambda x)$ we get $\|u_\lambda\|_{L^2_x} = \|u\|_{L^2_x}$ and

$$H(u_\lambda) \le \frac12\|\nabla_x u_\lambda\|^2_{L^2_x} = \frac{\lambda^2}{2}\|\nabla_x u\|^2_{L^2_x} \underset{\lambda\to 0}{\longrightarrow} 0.$$

Item d). For every (u,ψ) such that $\mathrm{supp}(u)\cap\mathrm{supp}(\sigma_1)$ and $\mathrm{supp}(\psi)\cap\mathrm{supp}(\sigma_1)\times\mathrm{supp}(\sigma_2)$ are non empty and for every $a\in\mathbb{R}$, we have

$$E(au, a|\psi|, 0) \,=\, a^2\left(\frac12\|\nabla_x u\|^2_{L^2_x} - a\int\left(\sigma_1\star\int\sigma_2|\psi|\,\mathrm{d}z\right)|u|^2\,\mathrm{d}x + \frac12\|\nabla_z|\psi|\|^2_{L^2_xL^2_z}\right) \underset{a\to+\infty}{\longrightarrow} -\infty$$

and $\|au\|^2_{L^2_x} = a^2\|u\|^2_{L^2_x}$. We conclude by using that $I_M \le 0$ and $M\mapsto I_M$ is non increasing.

Item e). We argue by contradiction: we suppose that $E(u,\psi,\chi) = I_M$ with $\|u\|^2_{L^2_x} = m$ and $0 < m < M$ (note that $I_M < 0$ implies $m \ne 0$). We first remark that $I_M < 0$ implies

$$\int\left(\sigma_1\star\int\sigma_2\psi\,\mathrm{d}z\right)|u|^2\,\mathrm{d}x < 0.$$

Then, by considering $v = (M/m)^{1/2}u$, $\varphi = (M/m)^{1/2}\psi$ and $\zeta = (M/m)^{1/2}\chi$ we get

$$\begin{aligned} I_M &\le E(v,\varphi,\zeta) \\ &= \frac{M}{m}\left(\frac12\|\nabla_x u\|^2_{L^2_x} + \underbrace{\sqrt{\frac{M}{m}}}_{>1}\underbrace{\int\left(\sigma_1\star\int\sigma_2\psi\,\mathrm{d}z\right)|u|^2\,\mathrm{d}x}_{<0} + \frac{1}{2c^2}\|\chi\|^2_{L^2_xL^2_z} + \frac12\|\nabla_z\psi\|^2_{L^2_xL^2_z}\right) \\ &\qquad\qquad\qquad < \frac{M}{m}E(u,\psi,\chi) = \frac{M}{m}I_M < I_M, \end{aligned}$$

which is a contradiction. Since (u,ψ,χ) is a minimizer of J_M, the Euler-Lagrange relations imply the existence of a Lagrange multiplier $\lambda_{u,\psi,\chi}$ such that $\nabla_{u,\psi,\chi}E(u,\psi,\chi) = \lambda_{u,\psi,\chi}\nabla_{u,\psi,\chi}(u\mapsto\|u\|^2_{L^2_x}) = 2\lambda_{u,\psi,\chi}(u,0,0)^t$. The first two components of this vectorial relation imply that (u,ψ) is a solution of (18a)–(18b) with $\omega = -\lambda_{u,\psi,\chi}$ and the third component implies that $\chi = 0$. Then $\psi = \Gamma\,\sigma_1\star|u|^2$ (which implies that $K_M = J_M$ is reached at u) and u is a solution of (19) with $\omega = -\lambda_{u,\psi,\chi}$. Moreover, by multiplying (19) by u and integrating over $\mathbb{R}^d$ we get

$$\frac12\|\nabla_x u\|^2_{L^2_x} + \omega\|u\|^2_{L^2_x} - \kappa\iint |u|^2(x)\Sigma(x-y)|u|^2(y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

It follows that

$$0 > J_M = K_M = \frac{1}{2}\|\nabla_x u\|^2_{L^2_x} - \frac{\kappa}{2}\iint |u|^2(x)\Sigma(x-y)|u|^2(y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= -\omega\|u\|^2_{L^2_x} + \frac{\kappa}{2}\iint |u|^2(x)\Sigma(x-y)|u|^2(y)\,\mathrm{d}x\,\mathrm{d}y$$

and thus $\omega > 0$. Eventually, thanks to the fact that ω is a positive number, one can prove by standard arguments that u is in the Schwartz class (we refer the reader to [20, Theorem 8] and its proof in [27, Remark 1]).

Item f). Let us denote by C the optimal constant of the homogeneous Sobolev embedding $\|f\|_{L^{2d/(d-2)}_x} \le C\|\nabla_x f\|_{L^2_x}$ (note that this estimate requires $d \ge 3$). Since $E(u, \Gamma\,\sigma_1 \star |u|^2, 0) = H(u)$ and by using the estimate

$$\iint |u|^2(x)\Sigma(x-y)|u|^2(y)\,\mathrm{d}x\,\mathrm{d}y \le \|\Sigma \star |u|^2\|_{L^\infty_x}\|u\|^2_{L^2_x}$$
$$\le \|\Sigma\|_{L^{\frac{d}{2}}_x}\|\,|u|^2\,\|_{L^{\frac{d}{d-2}}_x}\|u\|^2_{L^2_x} = \|\Sigma\|_{L^{\frac{d}{2}}_x}\|u\|^2_{L^{\frac{2d}{d-2}}_x}\|u\|^2_{L^2_x} \le C^2\|\Sigma\|_{L^{\frac{d}{2}}_x}\|\nabla_x u\|^2_{L^2_x}\|u\|^2_{L^2_x},$$

we eventually obtain

$$E(u, \Gamma\,\sigma_1 \star |u|^2, 0) \ge \frac{1}{2}\left(1 - \kappa C^2\|\Sigma\|_{L^{\frac{d}{2}}_x}\|u\|^2_{L^2_x}\right)\|\nabla_x u\|^2_{L^2_x},$$

and K_M is non negative as soon as $1 - \kappa C^2\|\Sigma\|_{L^{d/2}_x} M > 0$. The case of the dimension $d = 2$ can be treated as follows:

$$\iint |u|^2(x)\Sigma(x-y)|u|^2(y)\,\mathrm{d}x\,\mathrm{d}y \le \|\Sigma \star |u|^2\|_{L^2_x}\|\,|u|^2\|_{L^2_x}$$
$$\le \|\Sigma\|_{L^1_x}\|\,|u|^2\,\|_{L^2_x}\|\,|u|^2\|_{L^2_x} = \|\Sigma\|_{L^1_x}\|u\|^4_{L^4_x} \le \widetilde{C}^2\|\Sigma\|_{L^1_x}\|\nabla_x u\|^2_{L^2_x}\|u\|^2_{L^2_x},$$

where the last estimate is obtained thanks to the Gagliardo-Nirenberg inequality. ■

Thanks to the previous arguments, Theorem 2.1-(iii) follows from Theorem 2.1-(i): in the proof we will construct a minimizer such that u is non negative, radially symmetric and non increasing. We are thus left with the task of proving Theorem 2.1-(i).

Proof of Theorem 2.1-(i). We fix $M > 0$ and we consider a minimizing sequence $(u_\nu, \psi_\nu, \chi_\nu)_{\nu\in\mathbb{N}}$ of I_M. We start by constructing from this sequence another minimizing sequence with specific properties. Since $E(u_\nu, \psi_\nu, 0) \le E(u_\nu, \psi_\nu, \chi_\nu)$, we can take $\chi_\nu = 0$ for every ν. Moreover, owing to convexity properties, we have $E(|u_\nu|, -|\psi_\nu|, 0) \le E(u_\nu, \psi_\nu, 0)$ and we can suppose $u_\nu \ge 0$ and $\psi_\nu \le 0$. Finally, the density of linear combinations of tensor product in $L^2_x\dot{H}^1_z$ allows us to assume that every ψ_ν can be written as

$$\psi_\nu(x,z) = -\sum_{i=0}^{N_\nu} f_i^\nu(x) g_i^\nu(z),$$

where $f_i^\nu \in L^2_x$ and $g_i^\nu \in \dot{H}^1_z$ are positive functions. Possibly at the price of decomposing the g_i^ν's on a Hilbert basis of $\dot{H}^1_z$, we can suppose that for each ν, $(g_i^\nu)_{i\in\mathbb{N}}$ forms an orthogonal family and we obtain

$$E(u_\nu,\psi_\nu,0) = \frac{1}{2}\|\nabla_x u_\nu\|_{L^2_x}^2 - \sum_{i=0}^{N_\nu}\left(\int_{\mathbb{R}^n}\sigma_2(z)g_i^\nu(z)\,\mathrm{d}z\right)\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_\nu(x)|^2\sigma_1(x-y)f_i^\nu(y)\,\mathrm{d}x\,\mathrm{d}y\right) + \sum_{i=0}^{N_\nu}\|f_i^\nu\|_{L^2_x}^2\|g_i^\nu\|_{\dot{H}^1_z}^2.$$

From here we can apply the symmetric decreasing rearrangement theory in order to obtain, see [21, chapter 3], $\|u_\nu^*\|_{L^2_x}^2 = \|u_\nu\|_{L^2_x}^2$, $\|\nabla_x u_\nu^*\|_{L^2_x}^2 \le \|\nabla_x u_\nu\|_{L^2_x}^2$, $\|f_i^{\nu,*}\|_{L^2_x}^2 = \|f_i^\nu\|_{L^2_x}^2$ and

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_\nu(x)|^2\sigma_1(x-y)f_i^\nu(y)\,\mathrm{d}x\,\mathrm{d}y \le \iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_\nu^*(x)|^2\sigma_1^*(x-y)f_i^{\nu,*}(y)\,\mathrm{d}x\,\mathrm{d}y,$$

where $\cdot^*$ stands for the symmetric decreasing rearrangement of a given function. Since σ_1 is assumed non negative, radially symmetric and non increasing, $\sigma_1^* = \sigma_1$ and since

$$\sum_{i=0}^{N_\nu}\|f_i^{\nu,*}\|_{L^2_x}^2\|g_i^\nu\|_{\dot{H}^1_z}^2 = \left\|\sum_{i=0}^{N_\nu} f_i^{\nu,*}g_i^\nu\right\|_{L^2_x\dot{H}^1_z}^2,$$

we eventually obtain $E(u_\nu^*,\widetilde{\psi}_\nu,0) \le E(u_\nu,\psi_\nu,0)$, where $\widetilde{\psi}_\nu = \sum_{i=0}^{N_\nu} f_i^{\nu,*}g_i^\nu$. From now on, we will use the abuse of notation $u_\nu = u_\nu^*$ and $\psi_\nu = \widetilde{\psi}_\nu$.

Having disposed of these preliminaries, we enter into the heart of the proof. Thanks to (35) we know that $(u_\nu)_{\nu\in\mathbb{N}}$ is bounded in H^1_x and $(\psi_\nu)_{\nu\in\mathbb{N}}$ is bounded in $L^2_x\dot{H}^1_z$. Hence we can suppose, possibly at the price of extracting subsequences, that $(u_\nu)_{\nu\in\mathbb{N}}$ converges weakly to u in H^1_x, $(\psi_\nu)_{\nu\in\mathbb{N}}$ converges weakly to ψ in $L^2_x\dot{H}^1_z$. We have $\|u\|_{L^2_x}^2 \le M$, $\|\nabla_x u\|_{L^2_x}^2 \le \liminf_{\nu\to\infty}\|\nabla_x u_\nu\|_{L^2_x}^2$ and $\|\psi\|_{L^2_x\dot{H}^1_z}^2 \le \liminf_{\nu\to\infty}\|\psi_\nu\|_{L^2_x\dot{H}^1_z}^2$. In order to conclude the proof it only remains to prove that

$$\int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\psi_\nu\,\mathrm{d}z\right)|u_\nu(x)|^2\,\mathrm{d}x \underset{\nu\to+\infty}{\longrightarrow} \int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\psi\,\mathrm{d}z\right)|u(x)|^2\,\mathrm{d}x. \tag{36}$$

Indeed, (36) now implies $E(u,\psi,0) \le \liminf_{\nu\to\infty}E(u_\nu,\psi_\nu,0) = I_M$ and we eventually conclude that I_M is reached at $(u,\psi,0)$.

We turn to (36). On the one hand, in the case $d \ge 2$ we can use the symmetry property of the functions $u_\nu \in H^1_{rad}$ in order to justify the strong convergence of u_ν to u in L^p_x for $2 < p < p_c$ (where $p_c = 2d/(d-2)$ if $d \ge 3$ and $p_c = +\infty$ if $d = 2$), see [22, 36] for such compactness statements based on symmetry properties. On the other hand, in the case $d = 1$, by using a diagonal argument and extracting further subsequences if necessary, we know that $(u_\nu)_{\nu\in\mathbb{N}}$ converges also pointwise to u. Since for every ν, u_ν is a non negative even function with a non increasing profile, for almost every $x \in \mathbb{R}^d$ we get

$$2|x|\,|u_\nu(x)|^2 \le \int_{-|x|}^{|x|}|u_\nu(y)|^2\,\mathrm{d}y \le M \quad\text{and then}\quad |u_\nu(x)| \le \sqrt{\frac{M}{2|x|}} \lesssim |x|^{-1/2}.$$

Thanks to this uniform estimate with respect to ν, we can justify that the sequence $(|u_\nu|^p)_{\nu\in\mathbb{N}}$ is tight for every $2 < p < +\infty$. Combining this property with the compact embedding $H^1(\mathbb{R}_x) \to L^p_{loc}(\mathbb{R}_x)$ for every $1 \le p < +\infty$ allows us to justify that the sequence $(u_\nu)_{\nu\in\mathbb{N}}$ converges strongly to u in any L^p_x with $2 < p < +\infty$. We can now conclude the proof as follows:

$$\begin{aligned}\int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2 \psi_\nu \,\mathrm{d}z\right) |u_\nu|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^d} (\sigma_1 \star |u_\nu|^2) \left(\int_{\mathbb{R}^n} \sigma_2 \psi_\nu \,\mathrm{d}z\right) \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \left[(\sigma_1 \star |u_\nu|^2) - (\sigma_1 \star |u|^2)\right] \left(\int_{\mathbb{R}^n} \sigma_2 \psi_\nu \,\mathrm{d}z\right) \mathrm{d}x + \int_{\mathbb{R}^d} (\sigma_1 \star |u|^2) \left(\int_{\mathbb{R}^n} \sigma_2 \psi_\nu \,\mathrm{d}z\right) \mathrm{d}x,\end{aligned}$$

where

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} \left[(\sigma_1 \star |u_\nu|^2) - (\sigma_1 \star |u|^2)\right] \left(\int_{\mathbb{R}^n} \sigma_2 \psi_\nu \,\mathrm{d}z\right) \mathrm{d}x\right| \\ &\qquad\qquad \lesssim \left\|(\sigma_1 \star |u_\nu|^2) - (\sigma_1 \star |u|^2)\right\|_{L^2_x} \|\psi_\nu\|_{L^2_x \dot{H}^1_z}.\end{aligned}$$

Note that the weak convergence of ψ_ν to ψ in $L^2_x \dot{H}^1_z$ implies the convergence of the second term of the right hand side to $\int (\sigma_1 \star \int \sigma_2 \psi \,\mathrm{d}z) |u|^2 \,\mathrm{d}x$. Indeed

$$\begin{aligned}&\int_{\mathbb{R}^d} (\sigma_1 \star |u|^2) \left(\int_{\mathbb{R}^n} \sigma_2 \psi_\nu \,\mathrm{d}z\right) \mathrm{d}x \\ &\quad= \iint_{\mathbb{R}^d \times \mathbb{R}^n} (\sigma_1 \star |u|^2) \sigma_2 \,\psi_\nu \,\mathrm{d}x \,\mathrm{d}z = \iint_{\mathbb{R}^d \times \mathbb{R}^n} |\zeta| (\sigma_1 \star |u|^2)(x) \frac{\widehat{\sigma}_2(\zeta)}{|\zeta|^2} \,|\zeta| \overline{\widehat{\psi}_\nu(x,\zeta)} \,\mathrm{d}x \,\mathrm{d}\zeta \\ &\quad \underset{\nu \to +\infty}{\longrightarrow} \iint_{\mathbb{R}^d \times \mathbb{R}^n} |\zeta| (\sigma_1 \star |u|^2)(x) \frac{\widehat{\sigma}_2(\zeta)}{|\zeta|^2} \,\overline{\widehat{\psi}(x,\zeta)} \,\mathrm{d}x \,\mathrm{d}\zeta = \int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2 \psi \,\mathrm{d}z\right) |u|^2 \,\mathrm{d}x,\end{aligned}$$

where we used $n \geq 3$ in order to justify that $\zeta \mapsto \widehat{\sigma}_2(\zeta)/|\zeta|$ is an element of L^2_ζ. Thus, it only remains to prove that $\sigma_1 \star |u_\nu|^2$ converges strongly to $\sigma_1 \star |u|^2$ in L^2_x. To this end, we remark that

$$\sigma_1 \star |u_\nu|^2 - \sigma_1 \star |u|^2 = \sigma_1 \star \Big(|u_\nu - u + u|^2 - |u|^2 \Big) = \sigma_1 \star \Big(|u_\nu - u|^2 + 2\mathrm{Re}\,(u_\nu - u)\bar{u} \Big).$$

By using Young's inequalities we obtain for every $1 \leq p, q \leq +\infty$ with $1/p + 1/q = 1 + 1/2$

$$\begin{aligned}\left\|(\sigma_1 \star |u_\nu|^2) - (\sigma_1 \star |u|^2)\right\|_{L^2_x} &\leq \|\sigma_1\|_{L^p_x} \left\| |u_\nu - u|^2 + 2\mathrm{Re}\,(u_\nu - u)\bar{u} \right\|_{L^q_x} \\ &\leq \|\sigma_1\|_{L^p_x} \left(\|u_\nu - u\|^2_{L^{2q}_x} + 2\|u_\nu - u\|_{L^{2q}_x} \|u\|_{L^{2q}_x} \right).\end{aligned}$$

Then, since q can be chosen arbitrarily in $[1,2]$, we can always pick q such that $2q \in (2, p_c)$ and the strong convergence of u_ν to u in L^q_x for every $q \in (2, p_c)$ allows us to conclude. ■

Let us complete this Section with some comments on the uniqueness issue for the minimization problem J_M and complementary properties of the solutions. As soon as J_M is reached at (u, ψ, χ), we have $\chi = 0$, $\psi = \Gamma\, \sigma_1 \star |u|^2$ and $K_M = J_M$ is reached at u. Hence J_M admits a unique minimizer if and only if K_M admits a unique minimizer. In [20] E. Lieb fully answers the question of the uniqueness of the minimizer of K_M for the Newtonian kernel $\Sigma^0(x) = \frac{1}{|x|}$ in dimension $d = 3$. A first step of the proof consists in proving that if K_M is reached at u then, up to a translation and a change of phase, u is positive, radially symmetric and decreasing. The proof uses the fact that $r \mapsto 1/r$ is decreasing, see [20, Lemma 3 and Corollary 4]. Here, we suppose that σ_1 is non increasing (σ_1 strictly decreasing is not compatible with σ_1 compactly supported) and we cannot apply this reasoning. Nevertheless, the recent result of L. Ma-L. Zhao [28, Section 5] tells us that any non negative solution of (19) is strictly positive, radially symmetric and decreasing. This justifies that, if K_M is reached at u then, up to a translation and a change of phase, u is positive, radially symmetric and decreasing. The idea in [28] consists in writing (19) as a system

$$\Big(\omega - \frac{1}{2}\Delta\Big)Q = QX, \qquad X = \kappa\Sigma \star Q^2.$$

The operator $(\omega - \frac{1}{2}\Delta)$ is indeed invertible, and its inverse can be expressed by means of a convolution with the Bessel potential [35, Chapter V, Sect. 3]

$$\mathscr{J}(x) = \frac{1}{4\pi}\int_0^\infty e^{-\pi x^2/t} e^{-t/(4\pi)} t^{-(d-2)/2} \frac{\mathrm{d}t}{t}$$

(this kernel corresponds to the operator $(\mathbb{I}-\Delta)$). Therefore Q appears as the solution of an integral equation

$$Q = \mathscr{J} \star (QX), \qquad X = \kappa\Sigma \star Q^2.$$

The operator $(\omega - \frac{1}{2}\Delta)^{-1}$ is positive in the sense that the solution u of $(\omega - \frac{1}{2}\Delta)u = f$, with $f \geq 0$, $f \not\equiv 0$ is strictly positive. This reflects in the fact that $\mathscr{J}(x) > 0$ for any $x \in \mathbb{R}^d$. Since we already know that Q is non negative, we deduce that actually Q is positive. Moreover $\mathscr{J}$ is decreasing, Σ is non increasing, which allows us to adapt the moving plane strategy of [28]: we conclude that Q is radially symmetric, and monotone decreasing in the radial direction. The second step in Lieb's approach shows that K_M admits a unique positive, radially symmetric and decreasing minimizer [20, Theorem 10]. However, the proof relies strongly on the specific properties of the kernel $\Sigma^0(x) = 1/|x|$; the proof cannot be adapted to the present framework.

4 Orbital stability: concentration-compactness approach

Theorem 2.2 is a consequence of the following lemma.

Lemma 4.1 *Let $M > M_0$. If $(u_\nu, \psi_\nu, \chi_\nu)_{\nu\in\mathbb{N}} \subset H^1_x \times L^2_x \dot{H}^1_z \times L^2_x L^2_z$ is a sequence such that*

$$\|u_\nu\|^2_{L^2_x} \underset{\nu\to+\infty}{\longrightarrow} M \quad \textit{and} \quad E(u_\nu, \psi_\nu, \chi_\nu) \underset{\nu\to+\infty}{\longrightarrow} J_M,$$

then there exists a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ of elements of $\mathbb{R}^d$ and $(\widetilde{Q}, \widetilde{\Psi}) \in S_M$ such that, up to a subsequence,

$$\|u_\nu(\cdot - x_\nu) - \widetilde{Q}\|^2_{H^1_x} + \|\psi_\nu(\cdot - x_\nu, \cdot) - \widetilde{\Psi}\|^2_{L^2_x \dot{H}^1_z} + \|\chi_\nu\|^2_{L^2_x L^2_z} \underset{\nu\to+\infty}{\longrightarrow} 0.$$

Let us first explain how this lemma implies Theorem 2.2. We argue by contradiction. Let us assume the existence of $\varepsilon > 0$ and a sequence of initial data $(u_0^\nu, \psi_0^\nu, \chi_0^\nu)_{\nu\in\mathbb{N}}$ satisfying

$$\|u_0^\nu - Q\|^2_{H^1_x} + \|\psi_0^\nu - \Psi\|^2_{L^2_x \dot{H}^1_z} + \|\chi_0^\nu\|^2_{L^2_x L^2_z} \underset{\nu\to+\infty}{\longrightarrow} 0,$$

and such that for any $\nu \in \mathbb{N}$, the unique solution $(u^\nu, \psi^\nu, \chi^\nu)$ of (1a)-(1b) with initial data $(u_0^\nu, \psi_0^\nu, \chi_0^\nu)$ satisfies for some $t_\nu > 0$,

$$\inf_{(\widetilde{Q}, \widetilde{\Psi}) \in S_M} \Big(\|u^\nu(t_\nu) - \widetilde{Q}\|^2_{H^1_x} + \|\psi^\nu(t_\nu) - \widetilde{\Psi}\|^2_{L^2_x \dot{H}^1_z} + \|\chi^\nu(t_\nu)\|^2_{L^2_x L^2_z} \Big) > \varepsilon.$$

The strong convergence of u_0^ν to Q in H^1_x implies $\|u_0^\nu\|^2_{L^2_x} \to M$ while the continuity of the energy functional E with respect to $u \in H^1_x$, $\psi \in L^2_x \dot{H}^1_z$ and $\chi \in L^2_x L^2_z$ implies

$$E(u_0^\nu, \psi_0^\nu, \chi_0^\nu) \underset{\nu \to +\infty}{\longrightarrow} E(Q, \Psi, 0) = J_M.$$

By using the mass and energy conservations we check that the sequence $(u^\nu(t_\nu), \psi^\nu(t_\nu), \chi^\nu(t_\nu))_{\nu \in \mathbb{N}}$ fulfils the assumptions of Lemma 4.1 and we eventually obtain the required contradiction.

The proof of Lemma 4.1 is based on the concentration compactness lemma. In order to apply this lemma let us state and prove the following result on J_M.

Lemma 4.2 *(i) For every $M > M_0$ and for every $\theta > 1$, $J_{\theta M} < \theta J_M$.*
(ii) For every $M > M_0$ and for every $\alpha \in (0,1)$,

$$J_M < J_{\alpha M} + J_{(1-\alpha)M}. \tag{37}$$

Proof. *Item (i).* The proof follows the strategy of proof of Lemma 3.1-item e). Since $M > M_0$ there exists (u, ψ, χ) such that $\|u\|^2_{L^2_x} = M$ and $J_M = E(u, \psi, \chi)$. Hence, defining for $\theta > 1$, $v = \sqrt{\theta} u$, $\varphi = \sqrt{\theta}\psi$ and $\zeta = \sqrt{\theta}\chi$ and following the proof of Lemma 3.1-item e) we are led to

$$J_{\theta M} \leq E(v, \varphi, \zeta) < \theta E(u, \psi, \chi) = \theta J_M.$$

Item (ii). Let us distinguish two cases. The first case is $\alpha M \leq M_0$ or $(1-\alpha)M \leq M_0$. The case where these two conditions are satisfied is obvious:

$$J_M < 0 = J_{\alpha M} + J_{(1-\alpha)M}.$$

Hence let us assume, without loss of generality, that $\alpha M \leq M_0$ and $(1-\alpha)M > M_0$. Since $M \mapsto J_M$ is strictly decreasing on $(M_0, +\infty)$ (see the remarks after the statement of Lemma 3.1) we obtain

$$J_M < J_{(1-\alpha)M} = J_{\alpha M} + J_{(1-\alpha)M}.$$

The second case is $\alpha M > M_0$ and $(1-\alpha)M > M_0$. In this case we apply the previous item (with $\theta = 1/\alpha$ and $\theta = 1/(1-\alpha)$) as follows:

$$\begin{aligned} J_M = \alpha J_M + (1-\alpha) J_M &= \alpha J_{\frac{1}{\alpha}\alpha M} + (1-\alpha) J_{\frac{1}{1-\alpha}(1-\alpha)M} \\ &< \alpha \frac{1}{\alpha} J_{\alpha M} + (1-\alpha)\frac{1}{1-\alpha} J_{(1-\alpha)M} = J_{\alpha M} + J_{(1-\alpha)M}. \end{aligned}$$

■

Proof of Lemma 4.1. First of all, let us notice that we can consider, without loss of generality, that the sequence $(u_\nu)_{\nu \in \mathbb{N}}$ is such that for every $\nu \in \mathbb{N}$, $\|u_\nu\|^2_{L^2_x} = M$. Indeed, by considering the new sequence

$$\widetilde{u_\nu} = \frac{\sqrt{M}}{\|u_\nu\|_{L^2_x}} u_\nu,$$

and since $\|u_\nu\|^2_{L^2_x} \to M$ implies $\|\widetilde{u_\nu} - u_\nu\|_{H^1_x} \to 0$, if the conclusion of Lemma 4.1 holds for the sequence $(\widetilde{u_\nu})_{\nu \in \mathbb{N}}$, then it equally holds with the sequence $(u_\nu)_{\nu \in \mathbb{N}}$.

From now on we will consider that the sequence $(u_\nu)_{\nu \in \mathbb{N}}$ is such that $\|u_\nu\|^2_{L^2_x} = M$. Since $J_M \leq E(u_\nu, \psi_\nu, 0) \leq E(u_\nu, \psi_\nu, \chi_\nu)$ and $E(u_\nu, \psi_\nu, \chi_\nu) \to J_M$ when $\nu \to +\infty$ we obtain

$$\frac{1}{2c}\|\chi_\nu\|^2_{L^2_x L^2_z} = E(u_\nu, \psi_\nu, \chi_\nu) - E(u_\nu, \psi_\nu, 0) \underset{\nu \to +\infty}{\longrightarrow} 0.$$

Then, owing to (35), $(u_\nu)_{\nu\in\mathbb{N}}$ is bounded in H^1_x and $(\psi_\nu)_{\nu\in\mathbb{N}}$ is bounded in $L^2_x\dot{H}^1_z$. The concentration compactness lemma [23, 24] — here we use the version that can be found in [5, Prop. 1.7.6] — insures that there are only three different possible scenarii for the behavior of the sequence $(u_\nu)_{\nu\in\mathbb{N}}$.

Scenario 1: Evanescence. Up to a sub-sequence, for every $2 < q < 2^*$, $(u_\nu)_{\nu\in\mathbb{N}}$ converges strongly to 0 in L^q_x, where $2^* = +\infty$ if $d = 1$ or 2 and $2^* = 2d/(d-2)$ if $d \geq 3$. Let us assume $d \geq 3$; we have

$$\left|\int_{\mathbb{R}^d}\left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\psi_\nu \,\mathrm{d}z\right)|u_\nu|^2 \,\mathrm{d}x\right| \leq \left\|\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\psi_\nu \,\mathrm{d}z\right\|_{L^{d-1}_x} \| \, |u_\nu|^2\|_{L_x^{(d-1)/(d-2)}}$$
$$\leq \|\sigma_1\|_{L_x^{2(d-1)/(d+1)}}\|\sigma_2\|_{L_z^{2n/(n+2)}}\|\psi_\nu\|_{L^2_xL_z^{2n/(n-2)}} \lesssim \|\psi_\nu\|_{L^2_x\dot{H}^1_z}\|u_\nu\|^2_{L_x^{2(d-1)/(d-2)}}.$$

Since $(\psi_\nu)_{\nu\in\mathbb{N}}$ is bounded in $L^2_x\dot{H}^1_z$ and $2 < 2(d-1)/(d-2) < 2^*$, we eventually obtain

$$\int_{\mathbb{R}^d}\left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\psi_\nu \,\mathrm{d}z\right)|u_\nu|^2 \,\mathrm{d}x \underset{\nu\to+\infty}{\longrightarrow} 0.$$

Then

$$J_M = \lim_{\nu\to+\infty} E(u_\nu,\psi_\nu,0) = \lim_{\nu\to+\infty}\left(\frac{1}{2}\|\nabla_x u_\nu\|^2_{L^2_x} + \frac{1}{2}\|\nabla_z\psi_\nu\|^2_{L^2_xL^2_z}\right) \geq 0,$$

which contradicts $J_M < 0$.

Scenario 2: Dichotomy. Up to possible extraction, there exists two sequences $(v_\nu)_{\nu\in\mathbb{N}}$ and $(w_\nu)_{\nu\in\mathbb{N}}$, bounded in H^1_x and such that the following assertions hold

$$(i) \quad \exists\alpha \in (0,1) \text{ such that } \|v_\nu\|^2_{L^2_x} \underset{\nu\to+\infty}{\longrightarrow} \alpha M \text{ and } \|w_\nu\|^2_{L^2_x} \underset{\nu\to+\infty}{\longrightarrow} (1-\alpha)M,$$

$$(ii) \quad \forall\, 2 \leq q < 2^*, \qquad \|u_\nu\|^q_{L^q_x} - \|v_\nu\|^q_{L^q_x} - \|w_\nu\|^q_{L^q_x} \underset{\nu\to+\infty}{\longrightarrow} 0,$$

$$(iii) \quad \liminf_{\nu\to+\infty}\left(\|\nabla_x u_\nu\|^2_{L^2_x} - \|\nabla_x v_\nu\|^2_{L^2_x} - \|\nabla_x w_\nu\|^2_{L^2_x}\right) \geq 0.$$

With (ii), we infer

$$\left|\int_{\mathbb{R}^d}\left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\psi_\nu \,\mathrm{d}z\right)\left(|u_\nu|^2 - |v_\nu|^2 - |w_\nu|^2\right)\mathrm{d}x\right|$$
$$\leq \|\sigma_1\|_{L^2_x}\|\sigma_2\|_{L_z^{2n/(n+2)}}\|\psi_\nu\|_{L^2_x\dot{H}^1_z}\left(\int_{\mathbb{R}^d}\left||u_\nu|^2 - |v_\nu|^2 - |w_\nu|^2\right|\mathrm{d}x\right) \underset{\nu\to+\infty}{\longrightarrow} 0. \quad (38)$$

Note that we can apply (ii) because in the proof of the concentration compactness lemma [5] v_ν and w_ν are built in such way that $|u_\nu|^2 - |v_\nu|^2 - |w_\nu|^2 \geq 0$. Since

$$E(u_\nu,\psi_\nu,0) = \frac{1}{2}\left(\|\nabla_x u_\nu\|^2_{L^2_x} - \|\nabla_x v_\nu\|^2_{L^2_x} - \|\nabla_x w_\nu\|^2_{L^2_x}\right)$$
$$+ \int_{\mathbb{R}^d}\left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\psi_\nu \,\mathrm{d}z\right)\left(|u_\nu|^2 - |v_\nu|^2 - |w_\nu|^2\right)\mathrm{d}x$$
$$+ E(v_\nu,\psi_\nu,0) + E(w_\nu,\psi_\nu,0),$$

combining (38), (iii) and (i) yields

$$J_M = \lim_{\nu\to+\infty} E(u_\nu,\psi_\nu,0) \geq \liminf_{\nu\to+\infty}\left(E(v_\nu,\psi_\nu,0) + E(w_\nu,\psi_\nu,0)\right)$$
$$\geq \liminf_{\nu\to+\infty} E(v_\nu,\psi_\nu,0) + \liminf_{\nu\to+\infty} E(w_\nu,\psi_\nu,0) \geq J_{\alpha M} + J_{(1-\alpha)M},$$

which is a contradiction with (37), satisfied for $M \in (M_0, 2M_0)$.

Scenario 3: Compactness. Up to a sub-sequence, there exists a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{R}^d$ such that $v_\nu(x) = u_\nu(x - x_\nu)$ converges weakly to u in H^1_x and strongly to u in L^q_x for any $2 \le q < 2^*$. The sequence $\varphi_\nu(x,z) = \psi_\nu(x - x_\nu, z)$ is bounded in $L^2_x\dot{H}^1_z$ (note that $\|\varphi_\nu\|_{L^2_x\dot{H}^1_z} = \|\psi_\nu\|_{L^2_x\dot{H}^1_z}$) and then, up to a subsequence, $(\varphi_\nu)_{\nu\in\mathbb{N}}$ converges weakly to ψ in $L^2_x\dot{H}^1_z$. Since $(v_\nu)_{\nu\in\mathbb{N}}$ converges strongly to u in L^2_x we have $\|u\|^2_{L^2_x} = M$ and then $E(u,\psi,0) \ge J_M$. Moreover, reasoning as in (36) we get

$$\int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\varphi_\nu \,\mathrm{d}z\right) |v_\nu|^2 \,\mathrm{d}x \underset{\nu\to+\infty}{\longrightarrow} \int_{\mathbb{R}^d} \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2\psi \,\mathrm{d}z\right) |u|^2 \,\mathrm{d}x, \tag{39}$$

which allows us to justify that (u,ψ) lies in S_M:

$$\begin{aligned} J_M = \lim_{\nu\to+\infty} E(v_\nu,\varphi_\nu,0) &\ge \liminf_{\nu\to+\infty}\left(\frac{1}{2}\|\nabla_x v_\nu\|^2_{L^2_x}\right) \\ &+ \liminf_{\nu\to+\infty}\left(\int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\varphi_\nu\,\mathrm{d}z\right)|v_\nu|^2\,\mathrm{d}x\right) + \liminf_{\nu\to+\infty}\left(\frac{1}{2}\|\nabla_z\varphi_\nu\|^2_{L^2_xL^2_z}\right) \ge E(u,\psi,0). \end{aligned}$$

In order to conclude the proof it only remains to justify the strong convergence of $(v_\nu,\varphi_\nu)_{\nu\in\mathbb{N}}$ to (u,ψ) in $H^1_x \times L^2_x\dot{H}^1_z$. We already know that this convergence holds weakly. We combine

$$E(u,\psi,0) = J_M = \lim_{\nu\to+\infty} E(v_\nu,\varphi_\nu,0)$$

and (39) to deduce that

$$\frac{1}{2}\|\nabla_x v_\nu\|^2_{L^2_x} + \frac{1}{2}\|\nabla_z\varphi_\nu\|^2_{L^2_xL^2_z} \underset{\nu\to+\infty}{\longrightarrow} \frac{1}{2}\|\nabla_x u\|^2_{L^2_x} + \frac{1}{2}\|\nabla_z\psi\|^2_{L^2_xL^2_z},$$

holds, which allows us to conclude. ■

5 Strengthened orbital stability: approach by linearization

In this Section, we explain how Lemma 2.4 and Lemma 2.7 imply Theorem 2.8.

Step 1. The first step of the proof consists in checking that, up to the invariants of the equation, any $v \in H^1_x$ close enough to Q satisfies the orthogonality conditions (31a)–(31b). For that purpose, let us introduce the function $F : H^1_x \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ defined by

$$\begin{aligned} F_j\left(v,(y,\theta)\right) &= \left\langle \mathrm{Re}\, e^{-i\theta} v(\cdot+y)\ ,\ \partial_{x_j} Q \right\rangle_{L^2_x}, \qquad j = 1,\dots,d \\ F_{d+1}\left(v,(y,\theta)\right) &= \left\langle \mathrm{Im}\, e^{-i\theta} v(\cdot+y)\ ,\ Q \right\rangle_{H^1_x}. \end{aligned}$$

Direct computations show that $F\left(Q,(0,0)\right) = 0$ and $\mathrm{D}_{y,\theta} F\left(Q,(0,0)\right)$ is an invertible diagonal matrix (indeed $\partial_{y_j} F_j\left(Q,(0,0)\right) = \|\partial_{x_j} Q\|^2_{L^2_x}$ and $\partial_\theta F_{d+1}\left(Q,(0,0)\right) = -\|Q\|^2_{H^1_x}$). The implicit function theorem provides the existence of $\varepsilon_0 > 0$ and a C^1-diffeomorphism $G : B_{H^1_x}(Q, 2\varepsilon_0) \to U_{\varepsilon_0} \subset \mathbb{R}^{d+1}$, $G(v) = (x,\gamma)$ such that for every $v \in B_{H^1_x}(Q, 2\varepsilon_0)$ and every $(y,\theta) \in U_{\varepsilon_0}$, $F\left(v,(y,\theta)\right) = 0$ if and only if $(y,\theta) = G(v)$. Moreover, since

$$|(x,\gamma)| = |G(v) - G(Q)| \le \left(\sup \|\mathrm{D}_v G\|\right) \|v - Q\|_{H^1_x},$$

for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\eta(\varepsilon) > 0$ such that

$$\|v - Q, \varphi - \Psi\|_{\mathscr{H}}^2 + \frac{1}{c^2}\|\chi\|_{L_x^2 L_z^2}^2 \leq \eta(\varepsilon)^2$$

implies for $(x, \gamma) = G(v)$,

$$\left\|e^{-i\gamma} v(\cdot + x) - Q, \varphi(\cdot + x) - \Psi\right\|_{\mathscr{H}}^2 + \frac{1}{c^2}\|\chi\|_{L_x^2 L_z^2}^2 \leq \varepsilon^2.$$

Step 2. In this second step we show that, if for a given time $t_0 \in [0, +\infty)$, there exists $(x_0, \gamma_0) \in \mathbb{R}^{d+1}$ such that $v = e^{-i\gamma_0} u(t_0, \cdot + x_0)$ satisfies the orthogonality conditions (31a)–(31b) and the estimate

$$\left\|e^{-i\gamma_0} u(t_0, \cdot + x_0) - Q, \psi(t_0, \cdot + x_0) - \Psi\right\|_{\mathscr{H}}^2 + \frac{1}{c^2}\|\chi(t_0)\|_{L_x^2 L_z^2}^2 \leq \varepsilon^2 < \varepsilon_0^2,$$

then there exists a time $T^\star > t_0$ and two functions $x(t)$ and $\gamma(t)$ continuous in time such that $(x(t_0), \gamma(t_0)) = (x_0, \gamma_0)$ and, for every $t \in [t_0, T^\star)$,

i) $(x(t) - x_0, \gamma(t) - \gamma_0) \in U_{\varepsilon_0}$,

ii) $v = e^{-i\gamma(t)} u(t, \cdot + x(t))$ satisfies the orthogonality conditions (31a)–(31b),

iii) $\left\|e^{-i\gamma(t)} u(t, \cdot + x(t)) - Q, \psi(t, \cdot + x(t)) - \Psi\right\|_{\mathscr{H}}^2 + \dfrac{1}{c^2}\|\chi(t)\|_{L_x^2 L_z^2}^2 \leq \varepsilon^2.$

First, thanks to the time continuity of $t \mapsto (e^{-i\gamma_0} u(t, \cdot + x_0), \psi(t, \cdot + x_0)) \in \mathscr{H}$, there exists a time $T^\star > t_0$ such that for every $t \in [t_0, T^\star)$

$$\left\|e^{-i\gamma_0} u(t, \cdot + x_0) - Q, \psi(t, \cdot + x_0) - \Psi\right\|_{\mathscr{H}}^2 \leq 4\varepsilon^2 < 4\varepsilon_0^2.$$

Next, for every $t \in [t_0, T^\star)$ we can apply the first step to $v = e^{-i\gamma_0} u(t, \cdot + x_0)$ and we obtain the existence of $x(t)$ and $\gamma(t)$ such that $(x(t_0), \gamma(t_0)) = (x_0, \gamma_0)$ and such that i) and ii) hold. Moreover the continuity of $t \mapsto e^{-i\gamma_0} u(t, \cdot + x_0)$ implies the continuity of $t \mapsto x(t)$ and $t \mapsto \gamma(t)$. We notice also that we can extend by continuity $x(t)$ and $\gamma(t)$ at time $T^\star$ and this extension is such that $v = e^{-i\gamma(T^\star)} u(T^\star, \cdot + x(T^\star))$ still satisfies the orthogonality conditions (31a)–(31b).

We can now apply Lemma 2.4 and 2.7 as follows. Thanks to the conservation of mass and energy and to the invariance by translation and phase of these quantities we get

$$\begin{aligned} W(u_0, \psi_0, \chi_0) &= W(u(t), \psi(t), \chi(t)) \\ &= W\left(e^{-i\gamma(t)} u(t, \cdot + x(t)), \psi(t, \cdot + x(t)), \chi(t)\right) = W(Q + u^\varepsilon(t), \Psi + \psi^\varepsilon(t), \chi(t)), \end{aligned}$$

where

$$u^\varepsilon(t) = e^{-i\gamma(t)} u(t, \cdot + x(t)) - Q \qquad \text{and} \qquad \psi^\varepsilon(t) = \psi(t, \cdot + x(t)) - \Psi.$$

We make use of the decomposition (25) combined with Lemma 2.4 and 2.7; we obtain

$$\begin{aligned} &\bar{\nu}\|\operatorname{Re} u^\varepsilon, \psi^\varepsilon\|_{\mathscr{H}}^2 + \mu\|\operatorname{Im} u^\varepsilon\|_{H_x^1}^2 + \frac{1}{2c^2}\|\chi(t)\|_{L_x^2 L_z^2}^2 \\ &\qquad \leq W(u_0, \psi_0, \chi_0) - W(Q, \Psi, 0) + \frac{1}{\bar{\nu}}\left(\left|\langle \operatorname{Re} u^\varepsilon, Q\rangle_{L_x^2}\right|^2 + \sum_{j=1}^d \left|\langle \operatorname{Re} u^\varepsilon, \partial_{x_j} Q\rangle_{L_x^2}\right|^2\right) \\ &\qquad\qquad + \frac{1}{\mu}\left|\langle \operatorname{Im} u^\varepsilon, Q\rangle_{H_x^1}\right|^2 - \int_{\mathbb{R}^d}\left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2 \psi^\varepsilon(t)\, \mathrm{d}z\right)|u^\varepsilon(t)|^2\, \mathrm{d}x. \end{aligned}$$

Since $e^{-i\gamma(t)}u(t,\cdot+x(t))$ and Q satisfy the orthogonality conditions (31a)–(31b) we know that u^ε also satisfies these conditions. Moreover $\|Q\|_{L^2_x} = \|u(t)\|_{L^2_x} = \|u^\varepsilon + Q\|_{L^2_x}$ leads to

$$\|Q\|^2_{L^2_x} = \|u^\varepsilon\|^2_{L^2_x} + \|Q\|^2_{L^2_x} + 2\langle \operatorname{Re} u^\varepsilon, Q\rangle_{L^2_x} \quad \text{and then} \quad \langle \operatorname{Re} u^\varepsilon, Q\rangle_{L^2_x} = -\frac{1}{2}\|u^\varepsilon\|^2_{L^2_x},$$

which implies

$$\left|\langle \operatorname{Re} u^\varepsilon, Q\rangle_{L^2_x}\right|^2 \leq \frac{1}{4}\|u^\varepsilon\|^4_{L^2_x} \leq 4\,\varepsilon^4.$$

We also get

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\left(\sigma_1 \star \int_{\mathbb{R}^n}\sigma_2\psi^\varepsilon(t)\,\mathrm{d}z\right)|u^\varepsilon(t)|^2\,\mathrm{d}x\right| &\leq \|\sigma_1\|_{L^2_x}\|\sigma_2\|_{L_z^{2n/(n+2)}}\|\psi^\varepsilon(t)\|_{L^2_x\dot{H}^1_z}\|u^\varepsilon(t)\|^2_{L^2_x}\\ &\leq \|\sigma_1\|_{L^2_x}\|\sigma_2\|_{L_z^{2n/(n+2)}}\|u^\varepsilon(t),\psi^\varepsilon(t)\|^3_{\mathscr{H}} \leq 8\,\|\sigma_1\|_{L^2_x}\|\sigma_2\|_{L_z^{2n/(n+2)}}\,\varepsilon^3.\end{aligned}$$

Gathering these estimates leads eventually to (we recall that $W(u_0,\psi_0,\chi_0) - W(Q,\Psi,0) \leq \delta(\varepsilon)$)

$$\begin{aligned}&\|\operatorname{Re} u^\varepsilon,\psi^\varepsilon\|^2_{\mathscr{H}} + \|\operatorname{Im} u^\varepsilon\|^2_{H^1_x} + \frac{1}{c^2}\|\chi(t)\|^2_{L^2_xL^2_z}\\ &\qquad\leq \frac{1}{\min\left(\bar{\nu},\mu,\frac{1}{2}\right)}\left(\delta(\varepsilon) + \frac{4}{\bar{\nu}}\,\varepsilon^4 + 8\,\|\sigma_1\|_{L^2_x}\|\sigma_2\|_{L_z^{2n/(n+2)}}\,\varepsilon^3\right).\end{aligned}$$

By taking

$$\delta(\varepsilon) = \frac{\varepsilon^2}{2\,\min\left(\bar{\nu},\mu,\frac{1}{2}\right)},$$

and possibly at the price of picking a smaller ε_0, we eventually obtain iii) for every $t \in [t_0, T^\star]$.

Conclusion. We apply the first step with $v = u_0$, which insures the existence of $x(0)$ and $\gamma(0)$ such that we can apply the second step at time $t = 0$. Thus, since $T^\star > 0$ and since we took care to justify that the conclusions of second step is also valid at time $t = T^\star$, a classical argument on connected space allows us to conclude that $T^\star = +\infty$.

6 Coercivity of $\mathcal{L}_+$: proof of Lemma 2.7

This section is dedicated to the proof of Lemma 2.7, which is a key ingredient of the proof of Theorem 2.8. The kernel of $\mathcal{L}_+$ can be identified by using Lemma 2.5. Indeed, since $(f,\psi)^t \in \operatorname{Ker}(\mathcal{L}_+)$ implies

$$-\frac{1}{2}\Delta_z\psi + \sigma_2\left(\sigma_1 \star Qf\right) = 0,$$

we can express ψ in term of f as follows: $\psi = 2\Gamma\left(\sigma_1 \star Qf\right)$. Moreover the relation

$$\mathcal{L}_+\begin{pmatrix} f \\ 2\Gamma\left(\sigma_1 \star Qf\right)\end{pmatrix} = \begin{pmatrix} L_+ f \\ 0\end{pmatrix} \tag{40}$$

allows us to identify the kernel of $\mathcal{L}_+$ to the kernel of L_+: we eventually get

$$\operatorname{Ker}(\mathcal{L}_+) = \operatorname{Span}\{(\partial_{x_j}Q, \partial_{x_j}\Psi)^t,\ j = 1,\dots,d\}.$$

In order to prove the coercivity relations (30), we need the following two lemmas.

Lemma 6.1 *For every* $(f,\psi)\in\mathscr{H}$ *such that* $\langle f,Q\rangle_{L^2_x}=0$*, we have*

$$\left\langle \mathcal{L}_+\begin{pmatrix} f\\ \psi\end{pmatrix}\,,\,\begin{pmatrix} f\\ \psi\end{pmatrix}\right\rangle_{L^2_x\times L^2_xL^2_z}\ \geq\ 0.$$

Moreover, since $\mathrm{Ker}(\mathcal{L}_+)=\{(\partial_{x_j}Q,\partial_{x_j}\Psi)^t,\ j=1,\dots,d\}$ *and* $\langle\partial_{x_j}Q,Q\rangle_{L^2_x}=0$*, we know that this inequality cannot be strict.*

Lemma 6.2 *Let* $(f_\nu,\psi_\nu)_{\nu\in\mathbb{N}}$ *be a bounded sequence of* $\mathscr{H}$ *which converges weakly to* $(\bar f,\bar\psi)$ *in* $\mathscr{H}$*. Then, up to a sub-sequence if needed, we have the following two convergences:*

$$\int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\Psi\,\mathrm{d}z\right)|f_\nu|^2\,\mathrm{d}x\ \underset{\nu\to+\infty}{\longrightarrow}\ \int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\Psi\,\mathrm{d}z\right)|\bar f|^2\,\mathrm{d}x \tag{41}$$

and

$$\int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\psi_\nu\,\mathrm{d}z\right)Qf_\nu\,\mathrm{d}x\ \underset{\nu\to+\infty}{\longrightarrow}\ \int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\bar\psi\,\mathrm{d}z\right)Q\bar f\,\mathrm{d}x. \tag{42}$$

Proof of Lemma 6.1. Let f be a real valued function of H^1_x such that $\langle f,Q\rangle_{L^2_x}=0$, let ψ be a function of $L^2_x\dot H^1_z$ and let u be the function defined on $\mathbb{R}$ by

$$u(s)=\frac{\|Q\|_{L^2_x}}{\|Q+sf\|_{L^2_x}}(Q+sf).$$

One can check that $u(s)$ is a real valued function of H^1_x and $\|u(s)\|_{L^2_x}=\|Q\|_{L^2_x}$ for every $s\in\mathbb{R}$, u is smooth, $u(0)=Q$ and

$$u'(0)=f-\frac{\langle f,Q\rangle_{L^2_x}}{\|Q\|^2_{L^2_x}}Q=f.$$

Since $(Q,\Psi,0)$ is a minimizer of J_M, we know that for every $s\in\mathbb{R}$, $W(Q,\Psi,0)\leq W(u(s),\Psi+s\psi,0)$. Moreover (25) leads to

$$\begin{aligned}0\leq W(u(s),\Psi+s\psi,0)-W(Q,\Psi,0)&=\left\langle\mathcal{L}_+\begin{pmatrix}u(s)-Q\\ s\psi\end{pmatrix}\,,\,\begin{pmatrix}u(s)-Q\\ s\psi\end{pmatrix}\right\rangle_{L^2_x\times L^2_xL^2_z}\\ &\quad+\int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2 s\psi\,\mathrm{d}z\right)|u(s)-Q|^2\,\mathrm{d}x.\end{aligned}$$

Since $u(s)-Q=u(s)-u(0)=sf+o(s)$ (when s goes to 0), we eventually obtain

$$0\leq s^2\left\langle\mathcal{L}_+\begin{pmatrix}f\\ \psi\end{pmatrix}\,,\,\begin{pmatrix}f\\ \psi\end{pmatrix}\right\rangle_{L^2_x\times L^2_xL^2_z}+o(s^2),$$

which concludes the proof.

∎

Proof of Lemma 6.2. The proof uses in several places a basic result of integration theory, consequence of Egoroff's theorem [37, Proposition 3.9]: if a sequence $(g_\nu)_{\nu\in\mathbb{N}}\subset L^p(\mathbb{R}^d)$ converges weakly to some $\bar g$ in $L^p(\mathbb{R}^d)$ where $1\leq p<+\infty$ and if this sequence converges also a.e. to some g, then $\bar g=g$.

Here, the sequence $(f_\nu)_{\nu\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^d)$ and the compact embedding $H^1(\Omega)\to L^2(\Omega)$ which holds for any bounded open set $\Omega\subset\mathbb{R}^d$ implies that, up to a sub-sequence, $(f_\nu)_{\nu\in\mathbb{N}}$ converges strongly to $\bar f$ in $L^2(\Omega)$ and thus converges, up to a further sub-sequence, a.e. in Ω to $\bar f$. A diagonal argument yields the a.e. convergence of $(f_\nu)_{\nu\in\mathbb{N}}$ to $\bar f$ in $\mathbb{R}^d$. Moreover, by using the homogeneous Sobolev embedding in dimension $d=3$, the boundedness of $(f_\nu)_{\nu\in\mathbb{N}}$ in H^1_x implies its boundedness in L^2_x and L^6_x and, by interpolation, in any L^p_x with $2\le p\le 6$. Consequently, the sequence $(|f_\nu|^2)_{\nu\in\mathbb{N}}$ is bounded in L^3_x and, up to a sub-sequence, converges weakly in L^3_x to some g. Since this sequence converges also a.e. to $|\bar f|^2$, we have indeed $g=|\bar f|^2$.

To prove (41) we proceed as follows. Since $\Psi=\Gamma\,\sigma_1\star Q^2$ with Q lying in the Schwartz class, the weak convergence of $(|f_\nu|^2)_{n\in\mathbb{N}}$ to $|f|^2$ in L^3_x yields

$$\int\left(\sigma_1\star\int\sigma_2\Psi\,\mathrm{d}z\right)|f_\nu|^2\,\mathrm{d}x=-\kappa\int\left(\Sigma\star Q^2\right)|f_\nu|^2\,\mathrm{d}x$$
$$\underset{\nu\to+\infty}{\longrightarrow}-\kappa\int\left(\Sigma\star Q^2\right)|\bar f|^2\,\mathrm{d}x=\int\left(\sigma_1\star\int\sigma_2\Psi\,\mathrm{d}z\right)|\bar f|^2\,\mathrm{d}x.$$

We turn to (42). We split

$$\int\left(\sigma_1\star\int\sigma_2\psi_\nu\,\mathrm{d}z\right)Qf_\nu\,\mathrm{d}x=\iint\sigma_2\left(\sigma_1\star Qf_\nu\right)\psi_\nu\,\mathrm{d}x\,\mathrm{d}z$$
$$=\iint\sigma_2\left(\sigma_1\star Q(f_\nu-\bar f)\right)\psi_\nu\,\mathrm{d}x\,\mathrm{d}z+\iint\sigma_2\left(\sigma_1\star Q\bar f\right)\psi_\nu\,\mathrm{d}x\,\mathrm{d}z.$$

The weak convergence of $(\psi_\nu)_{\nu\in\mathbb{N}}$ to $\bar\psi$ in $L^2_x\dot H^1_z$ (note that σ_2 smooth and $n\ge 3$ imply $\sigma_2\in\dot H^{-1}_z$) directly implies that the second term of the right hand side converges to $\int(\sigma_1\star\int\sigma_2\bar\psi\,\mathrm{d}z)Q\bar f\,\mathrm{d}x$. It only remains to prove that the first term of the right hand side converges to 0. To this end, we are going to show that $(Qf_\nu)_{\nu\in\mathbb{N}}$ converges strongly to $Q\bar f$ in $L^{3/2}_x$. Indeed, $(|f_\nu|^{3/2})_{\nu\in\mathbb{N}}$ is bounded in L^2_x and, up to a sub-sequence it converges weakly to $g=|\bar f|^{3/2}$ in L^2_x. Since $Q^{3/2}\in L^2_x$, we get $\|Qf_\nu\|_{L^{3/2}_x}\to\|Q\bar f\|_{L^{3/2}_x}$ as $\nu\to\infty$. Moreover the sequence $(Qf_\nu)_{\nu\in\mathbb{N}}$ is also bounded in $L^{3/2}_x$ and, up to a further sub-sequence if needed, it converges weakly to $Q\bar f$ in $L^{3/2}_x$. Thus we get the announced strong convergence. We combine this strong convergence with the boundedness of $(\psi_\nu)_{n\in\mathbb{N}}$ in $L^2_x\dot H^1_z$ and we conclude as follows:

$$\left|\iint\sigma_2\left(\sigma_1\star Q(f_\nu-\bar f)\right)\psi_\nu\,\mathrm{d}x\,\mathrm{d}z\right|\le\|\sigma_2\|_{L^{2n/(n+2)}_z}\|\psi_\nu\|_{L^2_x\dot H^1_z}\|\sigma_1\star Q(f_\nu-\bar f)\|_{L^2_x}$$
$$\le\|\sigma_2\|_{L^{2n/(n+2)}_z}\|\psi_\nu\|_{L^2_x\dot H^1_z}\|\sigma_1\|_{L^{6/5}_x}\|Qf_\nu-Q\bar f\|_{L^{3/2}_x}\underset{\nu\to+\infty}{\longrightarrow}0.$$

■

We are now able to prove the coercivity relation (30).

Proof of (30)**.** We argue by contradiction, assuming the existence of a sequence of positive numbers $(\tilde\nu_k)_{k\in\mathbb{N}}$ which converges to 0 and the existence of a sequence $(f_k,\psi_k)_{k\in\mathbb{N}}$ in $\mathscr{H}$ such that for every k,

$$\left\langle\mathcal{L}_+\begin{pmatrix}f_k\\ \psi_k\end{pmatrix},\begin{pmatrix}f_k\\ \psi_k\end{pmatrix}\right\rangle_{L^2_x\times L^2_xL^2_z}<\tilde\nu_k\|f_k,\psi_k\|^2_{\mathscr{H}}-\frac{1}{\tilde\nu_k}\left(\left|\langle f_k,Q\rangle_{L^2_x}\right|^2+\sum_{j=1}^d\left|\langle f_k,\partial_{x_j}Q\rangle_{L^2_x}\right|^2\right).\qquad(43)$$

We can assume that $\|(f_k,\psi_k)\|_{\mathscr{H}}=1$ and thus, that there exists $\bar f\in H^1_x$ and $\bar\psi\in L^2_x\dot H^1_z$ such that $(f_k)_{k\in\mathbb{N}}$ converges weakly to $\bar f$ in H^1_x and $(\psi_k)_{k\in\mathbb{N}}$ converges weakly to $\bar\psi$ in $L^2_x\dot H^1_z$. On the one hand, thanks to the weak convergence of $(f_k)_{k\in\mathbb{N}}$, we get

$$\langle f_k, Q\rangle_{L^2_x} \underset{k\to+\infty}{\longrightarrow} \langle \bar f, Q\rangle_{L^2_x} \quad \text{and} \quad \langle f_k, \partial_{x_j} Q\rangle_{L^2_x} \underset{k\to+\infty}{\longrightarrow} \langle \bar f, \partial_{x_j} Q\rangle_{L^2_x},$$

while on the other hand (43) implies

$$0 \le \left|\langle f_k, Q\rangle_{L^2_x}\right|^2 + \sum_{j=1}^d \left|\langle f_k, \partial_{x_j} Q\rangle_{L^2_x}\right|^2 < \bar\nu_k^2 - \bar\nu_k \left\langle \mathcal{L}_+ \begin{pmatrix} f_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} f_k \\ \psi_k \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z} \underset{k\to+\infty}{\longrightarrow} 0,$$

bearing in mind that $\langle \mathcal{L}_+ h, h\rangle \le K\|h\|^2_{\mathscr{H}}$. We eventually obtain $\langle \bar f, Q\rangle_{L^2_x} = 0$ and $\langle \bar f, \partial_{x_j} Q\rangle_{L^2_x} = 0$. Knowing that $\bar f$ is orthogonal to Q, we can apply Lemma 6.1 in order to obtain

$$\left\langle \mathcal{L}_+ \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix}, \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z} \ge 0.$$

On the other hand, the relation

$$\begin{aligned}\left\langle \mathcal{L}_+ \begin{pmatrix} f_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} f_k \\ \psi_k \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z} &= \frac12 \|\nabla_x f_k\|^2_{L^2_x} + \omega\|f_k\|^2_{L^2_x} + \int \left(\sigma_1 \star \int \sigma_2 \Psi \,\mathrm{d}z\right) |f_k|^2 \,\mathrm{d}x \\ &\quad + 2\int \left(\sigma_1 \star \int \sigma_2 \psi_k \,\mathrm{d}z\right) Q f_k \,\mathrm{d}x + \frac12 \|\nabla_z \psi_k\|^2_{L^2_x L^2_z},\end{aligned}$$

coupled with Lemma 6.2 and (43) leads to

$$\begin{aligned}&\left\langle \mathcal{L}_+ \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix}, \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z} \\ &\quad \le \liminf_{k\to+\infty} \left\langle \mathcal{L}_+ \begin{pmatrix} f_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} f_k \\ \psi_k \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z} \le \limsup_{k\to+\infty} \left\langle \mathcal{L}_+ \begin{pmatrix} f_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} f_k \\ \psi_k \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z} \\ &\quad \le \limsup_{k\to+\infty} \left\{ \frac{1}{\bar\nu_k}\left(\left|\langle f_k, Q\rangle_{L^2_x}\right|^2 + \sum_{j=1}^d \left|\langle f_k, \partial_{x_j} Q\rangle_{L^2_x}\right|^2 \right) + \left\langle \mathcal{L}_+ \begin{pmatrix} f_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} f_k \\ \psi_k \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z} \right\} \\ &\quad \le \limsup_{k\to+\infty} \bar\nu_k = 0.\end{aligned}$$

We eventually deduce

$$\lim_{k\to+\infty} \left\langle \mathcal{L}_+ \begin{pmatrix} f_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} f_k \\ \psi_k \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z} = \left\langle \mathcal{L}_+ \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix}, \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z} = 0 \tag{44}$$

and thus $(\bar f, \bar\psi)$ is a minimizer of

$$\inf_{\langle f, Q\rangle_{L^2_x}=0} \left\langle \mathcal{L}_+ \begin{pmatrix} f \\ \psi \end{pmatrix}, \begin{pmatrix} f \\ \psi \end{pmatrix} \right\rangle_{L^2_x\times L^2_x L^2_z}. \tag{45}$$

We can now conclude as follows. First of all, the relation (44) coupled with Lemma 6.2 leads to the norm convergence

$$\frac12 \|\nabla_x f_k\|^2_{L^2_x} + \omega\|f_k\|^2_{L^2_x} + \frac12 \|\psi_k\|^2_{L^2_x \dot H^1_z} \underset{k\to+\infty}{\longrightarrow} \frac12 \|\nabla_x \bar f\|^2_{L^2_x} + \omega\|\bar f\|^2_{L^2_x} + \frac12 \|\bar\psi\|^2_{L^2_x \dot H^1_z}.$$

It implies the strong convergence of $(f_k, \psi_k)_{k\in\mathbb{N}}$ to $(\bar f, \bar\psi)$ in $\mathscr{H}$. In particular we know that $\|(\bar f, \bar\psi)\|_{\mathscr{H}} = 1$. Second of all, $(\bar f, \bar\psi)$ is a minimizer of (45) and the Euler Lagrange relation insures the existence of a real number λ such that

$$\mathcal{L}_+ \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix} = \lambda \begin{pmatrix} Q \\ 0 \end{pmatrix}.$$

The second component of this vectorial relation leads to $\bar\psi = 2\Gamma\left(\sigma_1 \star Q\bar f\right)$. From this relation we obtain the contradiction as follows: owing to (40), Lemma 2.5 and since $\bar f$ is orthogonal to Q and $\partial_{x_j} Q$, we get

$$\begin{aligned} 0 = \left\langle \mathcal{L}_+ \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix}, \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix} \right\rangle_{L^2_x \times L^2_x L^2_z} &= \left\langle \begin{pmatrix} L_+ \bar f \\ 0 \end{pmatrix}, \begin{pmatrix} \bar f \\ \bar\psi \end{pmatrix} \right\rangle_{L^2_x \times L^2_x L^2_z} \\ &= \left\langle L_+ \bar f, \bar f \right\rangle_{L^2_x} \geq \nu \|\bar f\|^2_{H^1_x} - \frac{1}{\nu}\left(\left|\langle \bar f, Q\rangle_{L^2_x}\right|^2 + \sum_{j=1}^d \left|\langle \bar f, \partial_{x_j} Q\rangle_{L^2_x}\right|^2 \right) = \nu \|\bar f\|^2_{H^1_x}. \end{aligned}$$

Thus $(\bar f, \bar\psi) = (0,0)$, which contradicts $\|\bar f, \bar\psi\|_{\mathscr{H}} = 1$. ■

7 Perturbation analysis: proof of Proposition 2.12

In this section, since there is no ambiguity, we will use the following shorthand notations, see Definition 2.11, $H^\varepsilon = H^{\Sigma^\varepsilon}$, $K^\varepsilon_M = K^{\Sigma^\varepsilon}_M$, $L^\varepsilon_+ = L_+(\Sigma^\varepsilon, Q^\varepsilon)$, $H^0 = H^{\Sigma^0}$, $K^0_M = K^{\Sigma^0}_M$ and $L^0_+ = L_+(\Sigma^0, Q^0)$. Before proving Proposition 2.12 let us check that $\sup(M^\varepsilon_0) < +\infty$. We remind the reader that the sequence of ground states $(Q^\varepsilon)_{\varepsilon>0}$ is well defined only if this supremum is finite.

Lemma 7.1 *Let **(H4)** be fulfilled. For every $M > 0$ there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, $M^\varepsilon_0 < M$.*

Proof. We start by showing that for every $u \in H^1_x$,

$$H^\varepsilon(u) \underset{\varepsilon\to 0}{\longrightarrow} H^0(u).$$

Indeed, thanks to the Cauchy-Schwarz inequality we have

$$\left| H^\varepsilon(u) - H^0(u) \right| = \left| \int |u|^2 \star (\Sigma^\varepsilon - \Sigma^0)(x)\, |u|^2(x)\, \mathrm{d}x \right| \leq \| |u|^2 \star (\Sigma^\varepsilon - \Sigma^0)\|_{L^\infty_x} \|u\|^2_{L^2_x},$$

and thanks to the homogeneous Sobolev embedding in dimension $d = 3$ we get

$$\begin{aligned} &\| |u|^2 \star (\Sigma^\varepsilon - \Sigma^0)\|_{L^\infty_x} \\ &\qquad \leq \|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x} \| |u|^2\|_{L^3_x} + \|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|>R}\|_{L^\infty_x} \| |u|^2\|_{L^1_x} \\ &\qquad\qquad \leq C\|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x} \|\nabla_x u\|^2_{L^2_x} + \|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|>R}\|_{L^\infty_x} \|u\|^2_{L^2_x}. \end{aligned}$$

Thus, assumption (32) leads to the required convergence. We conclude as follows. By using the results of E. Lieb in [20] we know that $K^0_M < 0$ is achieved at a unique positive and radially symmetric function Q^0. Then $H^\varepsilon(Q^0) \to H^0(Q^0) = K^0_M < 0$ implies $K^\varepsilon_M < 0$ as soon as ε is sufficiently small. Eventually Lemma 3.1-(d) and (e) allows us to conclude. ■

We turn to the proof of Proposition 2.12.

Proof of (i) Convergence. *Step 1.* We prove that for every $u \in H^1_x$ and for every $\delta, R > 0$, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$,

$$H^\varepsilon(u) \geq \frac{1}{2}\|\nabla_x u\|^2_{L^2_x} - \frac{\kappa C}{2}(\delta + cR)\|u\|^2_{L^2_x}\|\nabla_x u\|^2_{L^2_x} - \frac{\kappa}{2}\left(\delta + \frac{1}{R}\right)\|u\|^4_{L^2_x} \tag{46}$$

where C denotes the best constant in the homogeneous Sobolev embedding in dimension $d = 3$ and $c > 0$ is a constant. Since

$$\begin{aligned} H^\varepsilon(u) = \frac{1}{2}\|\nabla_x u\|^2_{L^2_x} - \frac{\kappa}{2}\iint |u|^2(x)\Sigma^\varepsilon(x-y)|u|^2(y)\,\mathrm{d}x\,\mathrm{d}y \\ \geq \frac{1}{2}\|\nabla_x u\|^2_{L^2_x} - \frac{\kappa}{2}\left|\iint |u|^2(x)\Sigma^\varepsilon(x-y)|u|^2(y)\,\mathrm{d}x\,\mathrm{d}y\right| \end{aligned}$$

we only have to estimate the last term of the right hand side. Again, we use the Cauchy-Schwarz inequality and the homogeneous Sobolev embedding and we obtain

$$\begin{aligned} \left|\iint |u|^2(x)\Sigma^\varepsilon(x-y)|u|^2(y)\,\mathrm{d}x\,\mathrm{d}y\right| &\leq C\|\Sigma^\varepsilon \mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x}\|u\|^2_{L^2_x}\|\nabla_x u\|^2_{L^2_x} + \|\Sigma^\varepsilon \mathbf{1}_{|x|>R}\|_{L^\infty_x}\|u\|^4_{L^2_x} \\ &\leq C\left(\|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x} + \|\Sigma^0 \mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x}\right)\|u\|^2_{L^2_x}\|\nabla_x u\|^2_{L^2_x} \\ &\quad + \left(\|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|>R}\|_{L^\infty_x} + \|\Sigma^0 \mathbf{1}_{|x|>R}\|_{L^\infty_x}\right)\|u\|^4_{L^2_x}. \end{aligned}$$

The quantities $\|\Sigma^0 \mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x}$ and $\|\Sigma^0 \mathbf{1}_{|x|>R}\|_{L^\infty_x}$ can be evaluated explicitly. Combined with the convergence (32), it allows us to obtain (46) for every $\delta > 0$ provided $\varepsilon > 0$ is sufficiently small.

Step 2. Estimate (46) has two consequences: firstly, the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is bounded in H^1_x and, secondly, the sequence $(K^\varepsilon_M)_{\varepsilon>0}$ is bounded from below (at least for $\varepsilon > 0$ sufficiently small) by $-\kappa(\delta + 1/R)M^2/2$. Indeed we already know that $\|Q^\varepsilon\|^2_{L^2_x} = M$ and for $\delta + cR > 0$ sufficiently small (that means $\varepsilon > 0$ is also sufficiently small), we have $\kappa C(\delta + cR)M/2 \leq 1/4$. Hence, (46) with $u = Q^\varepsilon$ becomes

$$H^\varepsilon(Q^\varepsilon) \geq \frac{1}{4}\|\nabla_x Q^\varepsilon\|^2_{L^2_x} - \frac{\kappa}{2}\left(\delta + \frac{1}{R}\right)M^2.$$

Since $H^\varepsilon(Q^\varepsilon) = K^\varepsilon_M < 0$ is negative for every $\varepsilon > 0$ we eventually deduce that $\|\nabla_x Q^\varepsilon\|_{L^2_x}$ is bounded. Moreover, it is clear that the sequence $(K^\varepsilon_M)_{\varepsilon>0}$ is bounded from below by $-\kappa(\delta + 1/R)M^2/2$, as soon as $\varepsilon > 0$ is sufficiently small.

Therefore, we know that $(Q^\varepsilon)_{\varepsilon>0}$ is bounded in H^1_x, and we also know the existence of two constant $a, A > 0$ such that for every $\varepsilon > 0$ sufficiently small, $-A \leq J^\varepsilon_M \leq -a$ (the existence of a comes from the proof of Lemma 7.1 where we proved that $K^\varepsilon_M \leq H^\varepsilon(Q^0) \to H^0(Q^0) = K^0_M < 0$). Moreover, since Q^ε is a solution of (19) with $\Sigma = \Sigma^\varepsilon$ and $\omega = \omega^\varepsilon$, by multiplying this equation by Q^ε and integrating over $\mathbb{R}^3$ we get

$$\omega^\varepsilon M = -\frac{1}{2}\|\nabla_x Q^\varepsilon\|^2_{L^2_x} + \kappa\iint |Q^\varepsilon|^2(x)\Sigma^\varepsilon(x-y)|Q^\varepsilon|^2(y)\,\mathrm{d}x\,\mathrm{d}y.$$

In turn, the sequence $(\omega^\varepsilon)_{\varepsilon>0}$ is bounded:

$$\begin{aligned} 0 < \frac{a}{M} \leq \omega^\varepsilon = -\frac{K^\varepsilon_M}{M} + \frac{\kappa}{2M}\iint |Q^\varepsilon|^2(x)\Sigma^\varepsilon(x-y)|Q^\varepsilon|^2(y)\,\mathrm{d}x\,\mathrm{d}y \\ \leq \frac{A}{M} + \frac{\kappa C}{2M}(\delta + cR)\|Q^\varepsilon\|^2_{L^2_x}\|\nabla_x Q^\varepsilon\|^2_{L^2_x} + \frac{\kappa}{2M}\left(\delta + \frac{1}{R}\right)\|Q^\varepsilon\|^4_{L^2_x}. \end{aligned}$$

There exists $\widetilde{Q} \in H^1_x$ and $\widetilde{\omega} > 0$ such that, up to a subsequence, $(Q^\varepsilon)_{\varepsilon>0}$ converges weakly to $\widetilde{Q}$ in H^1_x and $(\omega^\varepsilon)_{\varepsilon\geq 0}$ converges to $\widetilde{\omega}$. Since the functions Q^ε are positive and radially symmetric, we also know that $\widetilde{Q}$ is positive and radially symmetric, and $(Q^\varepsilon)_{\varepsilon>0}$ converges strongly to $\widetilde{Q}$ in L^p_x for $2 < p < 6$, see [22, 36] for such compactness statements based on symmetry properties.

Step 3. We are going to prove that $\widetilde{Q} = Q^0$ and $\widetilde{\omega} = \omega^0$. To this end, it is sufficient to prove that $\widetilde{Q}$ is a solution of the Choquard equation (19) with $\Sigma = \Sigma^0$, $\omega = \widetilde{\omega}$ and $\|\widetilde{Q}\|^2_{L^2_x} = M$. Indeed, we know that the Choquard equation with $\Sigma = \Sigma^0$ admits a unique positive, radially symmetric solution for $\omega = 1$ (see for instance [20] or [19]). This result can extended by a scaling argument for every $\omega > 0$. Hence, we can justify the following assertion: if two positive and radially symmetric solutions Q_1 and Q_2 of (19) with $\Sigma = \Sigma^0$, $\omega = \omega_1$ and $\omega = \omega_2$ have the same mass, then $Q_1 = Q_2$ and $\lambda_1 = \lambda_2$.

For every $\varepsilon > 0$ and for every $\varphi \in C^\infty_c(\mathbb{R}^3_x)$, we have

$$\frac{1}{2}\int \nabla_x Q^\varepsilon \cdot \nabla_x \varphi \,\mathrm{d}x + \omega^\varepsilon \int Q^\varepsilon \varphi \,\mathrm{d}x - \kappa \iint Q^\varepsilon \varphi(x) \Sigma^\varepsilon(x-y)|Q^\varepsilon|^2(y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

It is obvious that the first two terms converge respectively to $(\int \nabla_x \widetilde{Q} \cdot \nabla_x \varphi \,\mathrm{d}x)/2$ and $\widetilde{\omega}\int \widetilde{Q}\varphi\,\mathrm{d}x$ (note that for the second term we use the fact that $\|Q^\varepsilon\|_{L^2_x}$ is bounded with respect to ε). Let us now show that the third term converges to $-\kappa \iint \widetilde{Q}\varphi(x)\Sigma^0(x-y)|\widetilde{Q}|^2(y)\,\mathrm{d}x\,\mathrm{d}y$. For that purpose we decompose the difference as follows

$$\begin{aligned}
&\left| \iint Q^\varepsilon \varphi(x) \Sigma^\varepsilon(x-y)|Q^\varepsilon|^2(y)\,\mathrm{d}x\,\mathrm{d}y - \iint \widetilde{Q}\varphi(x)\Sigma^0(x-y)|\widetilde{Q}|^2(y)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\qquad \leq \underbrace{\left| \iint Q^\varepsilon\varphi(x)\left(\Sigma^\varepsilon(x-y) - \Sigma^0(x-y)\right)|Q^\varepsilon|^2(y)\,\mathrm{d}x\,\mathrm{d}y \right|}_{=I} \\
&\qquad\quad + \underbrace{\left| \iint \left(Q^\varepsilon(x) - \widetilde{Q}(x)\right)\varphi(x)\Sigma^0(x-y)|Q^\varepsilon|^2(y)\,\mathrm{d}x\,\mathrm{d}y \right|}_{=II} \\
&\qquad\quad + \underbrace{\left| \iint \widetilde{Q}\varphi(x)\Sigma^0(x-y)\left(|Q^\varepsilon|^2 - |Q^0|^2\right)(y)\,\mathrm{d}x\,\mathrm{d}y \right|}_{=III}.
\end{aligned}$$

The convergence of I follows from the boundedness of $(Q^\varepsilon)_{\varepsilon>0}$ in H^1_x together with the convergence (32):

$$\begin{aligned}
I &\leq \|Q^\varepsilon\varphi\|_{L^1_x}\|(\Sigma^\varepsilon - \Sigma^0)\star|Q^\varepsilon|^2\|_{L^\infty_x} \\
&\leq \|Q^\varepsilon\|_{L^2_x}\|\varphi\|_{L^2_x}\left(C\|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x}\|\nabla_x Q^\varepsilon\|^2_{L^2_x} + \|(\Sigma^\varepsilon - \Sigma^0)\mathbf{1}_{|x|>R}\|_{L^\infty_x}\|Q^\varepsilon\|^2_{L^2_x}\right).
\end{aligned}$$

The boundedness of $(Q^\varepsilon)_{\varepsilon>0}$ in L^2_x and the strong convergence of Q^ε to $\widetilde{Q}$ in L^p_x for $2 < p < 6$ with $p = 4$ and $p = 8/3$ imply the convergence of II (we use that $\Sigma^0\mathbf{1}_{|x|\leq R}$ lies in L^q_x for $1 \leq q < 3$ and $\Sigma^0\mathbf{1}_{|x|>R}$ lies in L^q_x for $q > 3$):

$$\begin{aligned}
II &\leq \|\Sigma^0 \star (Q^\varepsilon - \widetilde{Q})\varphi\|_{L^\infty_x}\|Q^\varepsilon\|^2_{L^2_x} \\
&\leq \left(\|\Sigma^0\mathbf{1}_{|x|\leq R}\|_{L^2_x}\|(Q^\varepsilon - \widetilde{Q})\varphi\|_{L^2_x} + \|\Sigma^0\mathbf{1}_{|x|>R}\|_{L^4_x}\|(Q^\varepsilon - \widetilde{Q})\varphi\|_{L^{4/3}_x}\right)\|Q^\varepsilon\|^2_{L^2_x} \\
&\leq \left(\|\Sigma^0\mathbf{1}_{|x|\leq R}\|_{L^2_x}\|Q^\varepsilon - \widetilde{Q}\|_{L^4_x}\|\varphi\|_{L^4_x} + \|\Sigma^0\mathbf{1}_{|x|>R}\|_{L^4_x}\|Q^\varepsilon - \widetilde{Q}\|_{L^{8/3}_x}\|\varphi\|_{L^{8/3}_x}\right)\|Q^\varepsilon\|^2_{L^2_x}.
\end{aligned}$$

For the last term we use almost the same strategy than for II. We write

$$
\begin{aligned}
III &\leq \|\widetilde{Q}\varphi\|_{L^1_x} \|\Sigma^0 \star (|Q^\varepsilon|^2 - |\widetilde{Q}|^2)\|_{L^\infty_x} \\
&\leq \|\widetilde{Q}\|_{L^2_x} \|\varphi\|_{L^2_x} \left(\left\|\Sigma^0 \mathbf{1}_{|x|\leq R}\right\|_{L^2_x} \left\| |Q^\varepsilon|^2 - |\widetilde{Q}|^2 \right\|_{L^2_x} + \left\|\Sigma^0 \mathbf{1}_{|x|>R}\right\|_{L^4_x} \left\| |Q^\varepsilon|^2 - |\widetilde{Q}|^2 \right\|_{L^{4/3}_x} \right).
\end{aligned}
$$

Since $|Q^\varepsilon|^2 - |\widetilde{Q}|^2 = |Q^\varepsilon - \widetilde{Q}|^2 + 2(Q^\varepsilon - \widetilde{Q})\widetilde{Q}$ we eventually obtain

$$
\begin{aligned}
\left\| |Q^\varepsilon|^2 - |\widetilde{Q}|^2 \right\|_{L^2_x} &\leq \left\| |Q^\varepsilon - \widetilde{Q}|^2 \right\|_{L^2_x} + 2 \left\| (Q^\varepsilon - \widetilde{Q})\widetilde{Q} \right\|_{L^2_x} \\
&\leq \left\| Q^\varepsilon - \widetilde{Q} \right\|^2_{L^4_x} + 2 \left\| Q^\varepsilon - \widetilde{Q} \right\|_{L^4_x} \left\| \widetilde{Q} \right\|_{L^4_x}
\end{aligned}
$$

and

$$
\begin{aligned}
\left\| |Q^\varepsilon|^2 - |\widetilde{Q}|^2 \right\|_{L^{4/3}_x} &\leq \left\| |Q^\varepsilon - \widetilde{Q}|^2 \right\|_{L^{4/3}_x} + 2 \left\| (Q^\varepsilon - \widetilde{Q})\widetilde{Q} \right\|_{L^{4/3}_x} \\
&\leq \left\| Q^\varepsilon - \widetilde{Q} \right\|^2_{L^{8/3}_x} + 2 \left\| Q^\varepsilon - \widetilde{Q} \right\|_{L^{8/3}_x} \left\| \widetilde{Q} \right\|_{L^{8/3}_x}.
\end{aligned}
$$

These convergences allow us to obtain that $\widetilde{Q}$ is a solution of (19) with $\Sigma = \Sigma^0$ and $\omega = \widetilde{\omega}$. It only remains to prove that $\|\widetilde{Q}\|^2_{L^2_x} = M$: the weak-$L^2_x$ convergence of Q^ε already implies $\|\widetilde{Q}\|^2_{L^2_x} \leq M$.

We multiply by Q^ε the Choquard equation satisfied by Q^ε and we integrate over $\mathbb{R}^3_x$; it yields

$$
-\omega^\varepsilon M = \frac{1}{2}\|\nabla_x Q^\varepsilon\|^2_{L^2_x} - \kappa \iint |Q^\varepsilon|^2(x)\Sigma^\varepsilon(x-y)|Q^\varepsilon|^2(y)\,\mathrm{d}x\,\mathrm{d}y.
$$

Taking $\liminf_{\varepsilon\to 0}$ leads to

$$
-\widetilde{\omega} M \geq \frac{1}{2}\|\nabla_x \widetilde{Q}\|^2_{L^2_x} - \kappa \limsup_{\varepsilon\to 0} \iint |Q^\varepsilon|^2(x)\Sigma^\varepsilon(x-y)|Q^\varepsilon|^2(y)\,\mathrm{d}x\,\mathrm{d}y.
$$

We justify as before that the last term converges to $\iint |\widetilde{Q}|^2(x)\Sigma^0(x-y)|\widetilde{Q}|^2(y)\,\mathrm{d}x\,\mathrm{d}y$. Since $\widetilde{Q}$ is a solution of (19) with $\Sigma = \Sigma^0$ and $\omega = \widetilde{\omega}$ we obtain

$$
-\widetilde{\omega} M \geq \frac{1}{2}\|\nabla_x \widetilde{Q}\|^2_{L^2_x} - \kappa \iint |\widetilde{Q}|^2(x)\Sigma^0(x-y)|\widetilde{Q}|^2(y)\,\mathrm{d}x\,\mathrm{d}y = -\widetilde{\omega}\|\widetilde{Q}\|^2_{L^2_x}.
$$

Since $\widetilde{\omega} > 0$, we eventually obtain $M \leq \|\bar{Q}\|^2_{L^2_x}$ and thus $\widetilde{Q} = Q^0$ and $\widetilde{\omega} = \omega^0$.

Step 5. In order to conclude the proof it only remains to justify that the weak convergence of (a sub-sequence of) $(Q^\varepsilon)_{\varepsilon>0}$ to Q^0 in H^1_x actually holds strongly (then, thanks to the uniqueness of Q^0, one can extend this convergence to the entire sequence). We already know that $\|Q^0\|^2_{L^2_x} = M = \|Q^\varepsilon\|^2_{L^2_x}$, which implies the strong convergence of $(Q^\varepsilon)_{\varepsilon>0}$ in L^2_x. We turn to the strong convergence of $(\nabla_x Q^\varepsilon)_{\varepsilon>0}$ in L^2_x. Thanks to the end of the previous step we have

$$
\lim_{\varepsilon\to 0}\|\nabla_x Q^\varepsilon\|^2_{L^2_x} = 2\left(-\omega^0 M + \kappa \iint |Q^0|^2(x)\Sigma^0(x-y)|Q^0|^2(y)\,\mathrm{d}x\,\mathrm{d}y\right) = \|\nabla_x Q^0\|^2_{L^2_x},
$$

which finishes the proof. ■

Proof of (ii) Coercivity. We fix $\varepsilon > 0$ and we consider a positive and radially symmetric minimizer Q^ε of K^ε_M. Proposition 2.5 gives

$$
\left\langle L^0_+ f, f\right\rangle_{L^2_x} \geq \nu^0\|f\|^2_{H^1_x} - \frac{1}{\nu^0}\left(\left|\langle f, Q^0\rangle_{L^2_x}\right|^2 + \sum_{j=1}^d \left|\langle f, \partial_{x_j} Q^0\rangle_{L^2_x}\right|^2\right).
$$

Next, we compute $\langle L_+^\varepsilon f, f\rangle$ as follows:

$$\begin{aligned}
\langle L_+^\varepsilon f, f\rangle_{L^2_x} &= \left\langle L_+^0 f, f\right\rangle_{L^2_x} + \left\langle (L_+^\varepsilon - L_+^0) f, f\right\rangle_{L^2_x}\\
&\geq \nu^0 \|f\|^2_{H^1_x} - \frac{1}{\nu^0}\left(\left|\langle f, Q^0\rangle_{L^2_x}\right|^2 + \sum_{j=1}^d \left|\langle f, \partial_{x_j} Q^0\rangle_{L^2_x}\right|^2\right) - \left|\left\langle (L_+^\varepsilon - L_+^0) f, f\right\rangle_{L^2_x}\right|\\
&\geq \nu^0 \|f\|^2_{H^1_x} - \frac{1}{\nu^0}\left(\left|\langle f, Q^\varepsilon\rangle_{L^2_x}\right|^2 + \sum_{j=1}^d \left|\langle f, \partial_{x_j} Q^\varepsilon\rangle_{L^2_x}\right|^2\right) - \frac{1}{\nu^0} R^\varepsilon - \left|\left\langle (L_+^\varepsilon - L_+^0) f, f\right\rangle_{L^2_x}\right|,
\end{aligned}$$

where

$$\begin{aligned}
R^\varepsilon = &\left|\langle f, Q^0 - Q^\varepsilon\rangle_{L^2_x}\right|^2 + \sum_{j=1}^d \left|\langle f, \partial_{x_j} Q^0 - \partial_{x_j} Q^\varepsilon\rangle_{L^2_x}\right|^2\\
&+ 2\left|\langle f, Q^0 - Q^\varepsilon\rangle_{L^2_x}\right|\left|\langle f, Q^\varepsilon\rangle_{L^2_x}\right| + 2\sum_{j=1}^d \left|\langle f, \partial_{x_j} Q^0 - \partial_{x_j} Q^\varepsilon\rangle_{L^2_x}\right|\left|\langle f, \partial_{x_j} Q^\varepsilon\rangle_{L^2_x}\right|.
\end{aligned}$$

Then we infer the following estimate: $R^\varepsilon \leq \alpha(Q^\varepsilon)\|f\|^2_{H^1_x}$ where $\alpha(Q) > 0$ and $\alpha(Q) \to 0$ when $\|Q - Q^0\|_{H^1_x} \to 0$. Moreover

$$\begin{aligned}
\left\langle (L_+^\varepsilon - L_+^0) f, f\right\rangle_{L^2_x} = &\left(\omega^\varepsilon - \omega^0\right)\|f\|^2_{L^2_x} - \kappa \int \left(\Sigma^\varepsilon \star |Q^\varepsilon|^2 - \Sigma^0 \star |Q^0|^2\right)|f|^2 \,\mathrm{d}x\\
&- 2\kappa \iint \left(Q^\varepsilon f(x)\Sigma^\varepsilon(x-y)Q^\varepsilon f(y) - Q^0 f(x)\Sigma^0(x-y)Q^0 f(y)\right) \mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

and from this expression we can obtain (thanks to a similar reasoning than in the proof of point (i)) the following estimate

$$\left|\left\langle (L_+^\varepsilon - L_+^0) f, f\right\rangle_{L^2_x}\right| \leq \beta(\Sigma^\varepsilon, Q^\varepsilon, \omega^\varepsilon)\|f\|^2_{H^1_x},$$

where $\beta(\Sigma, Q, \omega) > 0$ and $\beta(\Sigma, Q, \omega) \to 0$ when

$$\|(\Sigma - \Sigma^0)\mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x} + \|(\Sigma - \Sigma^0)\mathbf{1}_{|x|>R}\|_{L^\infty_x} + \|Q - Q^0\|_{H^1_x} + |\omega - \omega^0| \to 0.$$

This assertion applies for any $R > 0$; here R is fixed once for all (not necessarily small as in the proof of convergence). Gathering these two estimates leads to

$$\langle L_+^\varepsilon f, f\rangle_{L^2_x} \geq \left(\nu^0 - \frac{\alpha(Q^\varepsilon)}{\nu^0} - \beta(\Sigma^\varepsilon, Q^\varepsilon, \omega^\varepsilon)\right)\|f\|^2_{H^1_x} - \frac{1}{\nu^0}\left(\left|\langle f, Q^\varepsilon\rangle_{L^2_x}\right|^2 + \sum_{j=1}^d \left|\langle f, \partial_{x_j} Q^\varepsilon\rangle_{L^2_x}\right|^2\right).$$

The announced coercivity property holds for the ground state Q^ε provided $\alpha(Q^\varepsilon)/\nu^0 + \beta(\Sigma^\varepsilon, Q^\varepsilon, \omega^\varepsilon) < \nu^0$. Since $\alpha(Q)$ and $\beta(\Sigma, Q, \omega)$ converge to zero when $\|(\Sigma-\Sigma^0)\mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x} + \|(\Sigma-\Sigma^0)\mathbf{1}_{|x|>R}\|_{L^\infty_x} + \|Q-Q^0\|_{H^1_x} + |\omega-\omega^0| \to 0$, there exists $\delta > 0$ such that $\|(\Sigma-\Sigma^0)\mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x} + \|(\Sigma-\Sigma^0)\mathbf{1}_{|x|>R}\|_{L^\infty_x} + \|Q - Q^0\|_{H^1_x} + |\omega - \omega^0| < \delta$ implies $\alpha(Q)/\nu^0 + \beta(\Sigma, Q, \omega) < \nu^0$. Thanks to **(H4)** we can find $\bar{\varepsilon}_0 > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon}_0)$,

$$\|(\Sigma - \Sigma^0)\mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x} + \|(\Sigma - \Sigma^0)\mathbf{1}_{|x|>R}\|_{L^\infty_x} < \frac{\delta}{2}.$$

Therefore, possibly by choosing a smaller $\bar{\varepsilon}_0$ if necessary, for every $\varepsilon \in (0, \bar{\varepsilon}_0)$ and every positive and radially symmetric minimizer Q^ε of K_M^ε, we get

$$\|Q^\varepsilon - Q^0\|_{H^1_x} + |\omega^\varepsilon - \omega^0| < \frac{\delta}{2}.$$

We argue by contradiction to justify this. If this were not the case there would be a sequence $\varepsilon_k \to 0$ and a sequence of positive and radially symmetric minimizer $(Q^{\varepsilon_k})_{n\in\mathbb{N}}$ such that for every n,

$$\|Q^{\varepsilon_k} - Q^0\|_{H^1_x} + |\omega^{\varepsilon_k} - \omega^0| \geq \frac{\delta}{2}.$$

However we can apply point (i) to this sequence which insures that

$$\|Q^{\varepsilon_k} - Q^0\|_{H^1_x} + |\omega^{\varepsilon_k} - \omega^0| \underset{k\to+\infty}{\longrightarrow} 0,$$

a contradiction. ∎

8 Admissible form functions: proof of Proposition 2.10

The general strategy relies on the application of Proposition 2.12; hence we have to construct a sequence of potentials $(\Sigma^\varepsilon)_{\varepsilon>0}$, with the specific form $\Sigma^\varepsilon = \sigma_1^\varepsilon \star \sigma_1^\varepsilon$, which converges to Σ^0 in the sense of (32). This requires some care beyond the classical "regularization and truncature" approach. A similar difficulty arises, but in a different manner, when justifying the asymptotic regime of the Vlasov-Wave system (6a), (7) towards the Vlasov-Poisson equation [8]. The following simple examples are quite illuminating on the strategy.

Toy example 1. Let $\chi : \mathbb{R}^d \to [0,1]$ be a C_c^∞ function which satisfies $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Let

$$\Sigma^\varepsilon(x) = \frac{\chi(\varepsilon x)}{|x|}.$$

The analysis of this kernel is simple: due to the scale invariance of $\frac{1}{|x|}$, we have

$$\Sigma^\varepsilon(x) = \varepsilon \frac{\chi(\varepsilon x)}{|\varepsilon x|} = \varepsilon \Sigma^1(\varepsilon x).$$

As a matter of fact, we have

i) $H^{\Sigma^\varepsilon}(u) = \varepsilon^3 H^{\Sigma^1}(u^\varepsilon)$ where $u^\varepsilon(x) = \varepsilon^{-2} u(\varepsilon^{-1}x)$,

ii) Q^ε is a minimizer of $K_M^{\Sigma^\varepsilon} \iff Q(x) = \varepsilon^{-2} Q^\varepsilon(\varepsilon^{-1}x)$ is a minimizer of $K_{\varepsilon^{-1}M}^{\Sigma^1}$,

iii) $K_M^{\Sigma^\varepsilon} = \varepsilon^3 K_{\varepsilon^{-1}M}^{\Sigma^1}$,

iv) if Q^ε is a minimizer of $K_M^{\Sigma^\varepsilon}$, then $\omega(\Sigma^\varepsilon, Q^\varepsilon) = \varepsilon^2 \omega(\Sigma^1, Q)$ where $Q(x) = \varepsilon^{-2} Q^\varepsilon(\varepsilon^{-1}x)$,

v) $\langle L_+(\Sigma^\varepsilon, Q^\varepsilon) f^\varepsilon, f^\varepsilon \rangle_{L^2_x} = \varepsilon^3 \langle L_+(\Sigma^1, Q) f, f \rangle_{L^2_x}$ where $f(x) = \varepsilon^{-2} f^\varepsilon(\varepsilon^{-1}x)$ and still $Q(x) = \varepsilon^{-2} Q^\varepsilon(\varepsilon^{-1}x)$.

These relations proviode several useful information. For example, since for any fixed $\varepsilon > 0$, Σ^ε lies in $L^{3/2}_x$, Lemma 3.1 applies and justifies the existence of the mass threshold $M_0^{\Sigma^\varepsilon}$, which, in turn, can be expressed by means of $M_0^{\Sigma^1}$: $M_0^{\Sigma^\varepsilon} = \varepsilon M_0^{\Sigma^1} \to 0$. Furthermore, Σ^ε converges to Σ^0 in the sense of (32), and the conclusions of Proposition 2.12 hold. Then, relation v) allows us to extend the coercivity estimate to any radially symmetric minimizer of $K_m^{\Sigma^1}$ associated to a mass m larger than $M/\bar{\varepsilon}_0$, as illustrated by Fig. 2. Indeed ii), v) and Proposition 2.12-(ii) yield

$$\begin{aligned}
\left\langle L_+(\Sigma^1, Q)f, f\right\rangle_{L^2_x} &= \varepsilon^{-3} \left\langle L_+(\Sigma^\varepsilon, Q^\varepsilon)f^\varepsilon, f^\varepsilon\right\rangle_{L^2_x} \\
&\geq \varepsilon^{-3}\nu^\varepsilon \|f^\varepsilon\|^2_{H^1_x} - \frac{\varepsilon^{-3}}{\nu^0}\left(\left|\langle f^\varepsilon, Q^\varepsilon\rangle_{L^2_x}\right|^2 + \sum_{j=1}^3 \left|\langle f^\varepsilon, \partial_{x_j} Q^\varepsilon\rangle_{L^2_x}\right|^2\right) \\
&= \nu^\varepsilon \|\nabla_x f\|^2_{L^2_x} + \varepsilon^{-2}\nu^\varepsilon\|f\|^2_{L^2_x} - \frac{1}{\nu^0}\left(\varepsilon^{-2}\left|\langle f, Q\rangle_{L^2_x}\right|^2 + \varepsilon^{-1}\sum_{j=1}^3 \left|\langle f^\varepsilon, \partial_{x_j} Q^\varepsilon\rangle_{L^2_x}\right|^2\right)
\end{aligned}$$

which implies the announced coercivity property.

This example can be compared to the case of the Yukawa potential seen as a perturbation of the Newtonian potential in [17].

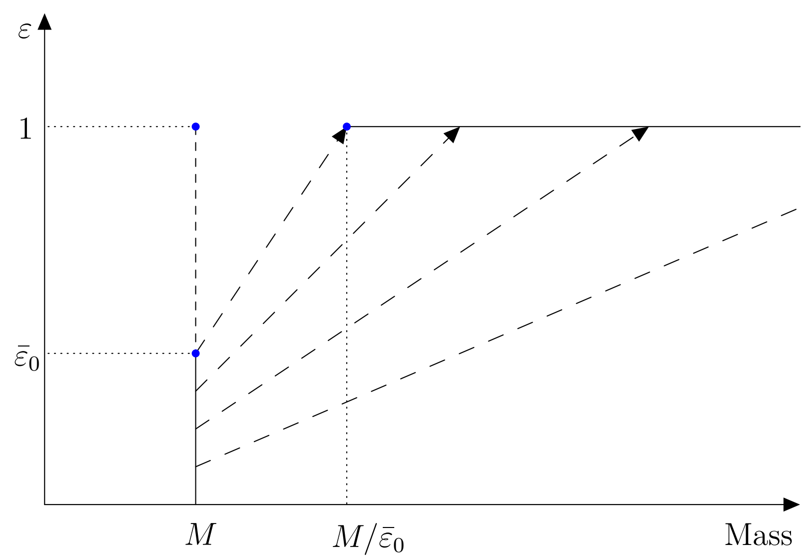

Figure 2: Illustration of the strategy: for the given mass M, the stability of the ground states is proved for the potentials Σ^ε, with $0 \leq \varepsilon < \bar{\varepsilon}_0$. By rescaling, we can go back to the potentials Σ^1, and ground states with a mass larger that $M/\bar{\varepsilon}_0$ are stable.

Toy example 2. Let $\alpha : \mathbb{R}^d \to [0, \infty)$ be a C^∞ function such that $\int \alpha \, \mathrm{d}x = 1$. We consider

$$\Sigma^\varepsilon(x) = \varepsilon^{-3} \int \frac{\alpha(\varepsilon^{-1}y)}{|x-y|} \, \mathrm{d}y.$$

Now, we have the scaling relation: $\Sigma^\varepsilon(x) = \varepsilon^{-1}\Sigma^1(\varepsilon^{-1}x)$, where

$$\Sigma^1(x) = \int \frac{\alpha(y)}{|x-y|}\,\mathrm{d}y.$$

We deduce that

$$Q^\varepsilon \text{ is a minimizer of } K_M^{\Sigma^\varepsilon} \iff Q(x) = \varepsilon^2 Q^\varepsilon(\varepsilon x) \text{ is a minimizer of } K_{\varepsilon M}^{\Sigma^1}.$$

Reasoning as in the previous example, we obtain that, for M sufficiently small, every positive and radially symmetric minimizer of $K_M^{\Sigma^1}$ satisfies the coercivity relation (29). In particular there is no mass threshold: $M_0^{\Sigma^1} = 0$. Since $\Sigma^1 \notin L_x^{3/2}$, this is not a contradiction with Lemma 3.1.

Toy example 3. We go back to the case of the dimension $d = 1$. In this case for any σ_1 satisfying **(H2)-(H3)** we consider the sequence of potential (Σ^ε) defined by

$$\Sigma^\varepsilon(x) = \varepsilon^{-1}\Sigma(\varepsilon^{-1}x), \qquad \Sigma = \sigma_1 \star \sigma_1.$$

Hence we obtain the equivalence

$$Q^\varepsilon \text{ is a minimizer of } K_M^{\Sigma^\varepsilon} \iff Q(x) = \varepsilon Q^\varepsilon(\varepsilon x) \text{ is a minimizer of } K_{\varepsilon M}^{\Sigma^1}.$$

Reasoning as above, we justify the existence of some $M^* > 0$ such that for every $M \in (0, M^*)$, every positive and even minimizer of $K_M^{\Sigma^1}$ satisfies the coercivity relation (29).

Main strategy. The toy examples 1 and 2 do not fit with our framework, where we are dealing with smooth and compactly supported potentials Σ. Then, in order to handle such a potential, the idea is (as usual) to combine the truncature and the regularization by setting

$$\Sigma^\varepsilon(x) = \varepsilon^{-3}\chi(\varepsilon x)\int \frac{\alpha(\varepsilon^{-1}y)}{|x-y|}\,\mathrm{d}y.$$

However, the scaling for the truncature and for the regularization are not the same, and the properties deduced from the scale invariance of $\frac{1}{|x|}$ break down. Instead, we consider a doubly indexed sequence of potentials

$$\Sigma^{\lambda,\mu}(x) = \lambda^{-3}\chi(\mu x)\int \frac{\alpha(\lambda^{-1}y)}{|x-y|}\,\mathrm{d}y$$

with $\lambda, \mu > 0$. We also introduce

$$\widetilde{\Sigma}^\epsilon(x) = \epsilon^{-3}\chi(x)\int \frac{\alpha(\epsilon^{-1}y)}{|x-y|}\,\mathrm{d}y.$$

We have the scaling relation $\Sigma^{\lambda,\mu}(x) = \mu\widetilde{\Sigma}^{\lambda\mu}(\mu x)$ which leads to the following lemma.

Lemma 8.1 *The following assertions hold:*

i) $H^{\Sigma^{\lambda,\mu}}(u) = \mu^3 H^{\widetilde{\Sigma}^\epsilon}(u^\mu)$ *where* $u^\mu(x) = \mu^{-2}u(\mu^{-1}x)$ *and* $\epsilon = \lambda\mu$, *ii)* $Q^{\lambda,\mu}$ *is a minimizer of* $K_M^{\widetilde{\Sigma}^{\lambda,\mu}} \iff Q(x) = \mu^{-2}Q^{\lambda,\mu}(\mu^{-1}x)$ *is a minimizer of* $K_{\mu^{-1}M}^{\widetilde{\Sigma}^{\epsilon}}$ *with* $\epsilon = \lambda\mu$,

iii) $K_M^{\Sigma^{\lambda,\mu}} = \mu^3 K_{\mu^{-1}M}^{\widetilde{\Sigma}^{\epsilon}}$ *with* $\epsilon = \lambda\mu$,

iv) if $Q^{\lambda,\mu}$ *is a minimizer of* $K_M^{\Sigma^{\lambda,\mu}}$*, then* $\omega(\Sigma^{\lambda,\mu}, Q^{\lambda,\mu}) = \mu^2\omega(\widetilde{\Sigma}^{\epsilon}, Q)$ *where* $Q(x) = \mu^{-2}Q^{\lambda,\mu}(\mu^{-1}x)$ *and* $\epsilon = \lambda\mu$,

v) $\left\langle L_+(\Sigma^{\lambda,\mu}, Q^{\lambda,\mu})f^{\lambda,\mu}, f^{\lambda,\mu}\right\rangle_{L^2_x} = \mu^3 \left\langle L_+(\widetilde{\Sigma}^{\epsilon}, Q)f, f\right\rangle_{L^2_x}$ *where* $Q(x) = \mu^{-2}Q^{\lambda,\mu}(\mu^{-1}x)$, $f(x) = \mu^{-2}f^{\lambda,\mu}(\mu^{-1}x)$ *and* $\epsilon = \lambda\mu$.

Let us suppose for a while that the sequence $(\Sigma^{\lambda,\mu})_{\lambda,\mu>0}$ converges to Σ^0 in the sense of (32) as λ and μ tend to 0. Then there exists $\lambda_0 > 0$ and $\mu_0 > 0$ such that for any $(\lambda, \mu) \in (0, \lambda_0) \times (0, \mu_0)$, the conclusions of Proposition 2.12 hold. Based on Lemma 8.1, we infer the following statement.

Proposition 8.2 (i) *For every* $(\lambda, \mu) \in (0, \lambda_0) \times (0, \mu_0)$ *and for every positive and radially symmetric minimizer* Q *of* $K_{\mu^{-1}M}^{\widetilde{\Sigma}^{\epsilon}}$ *with* $\epsilon = \lambda\mu$*, the operator* $L_+(\widetilde{\Sigma}^{\epsilon}, Q)$ *satisfies Lemma 2.5.*
(ii) *In particular, for* $\epsilon \in (0, \lambda_0\mu_0)$ *fixed, applying (i) to any* $(\lambda, \mu) \in (0, \lambda_0)\times(0, \mu_0)$ *such that* $\lambda\mu = \epsilon$ *implies that for any* $m \in (\mu_0^{-1}M, \lambda_0\epsilon^{-1}M)$ *and any positive and radially symmetric minimizer* Q *of* $K_m^{\widetilde{\Sigma}^{\epsilon}}$*, the operator* $L_+(\widetilde{\Sigma}^{\epsilon}, Q)$ *satisfies Lemma 2.5.*

Item *(ii)* implies, up to the fact that $\widetilde{\Sigma}^{\epsilon}$ can be cast under the form $\widetilde{\Sigma}^{\epsilon} = \widetilde{\sigma_1}^{\epsilon} \star \widetilde{\sigma_1}^{\epsilon}$, that the set of admissible form function $\mathscr{A}$ is non empty. Then, to conclude the proof it only remains to slightly adapt the previous construction in order to obtain a sequence $\Sigma^{\lambda,\mu}$ satisfying **(H4)**. We proceed as follows. Let α, χ be two $C_c^\infty(\mathbb{R}^3)$, non negative, radially symmetric, compactly supported and non increasing functions, with $\chi(x) = 1$ in a neighborhood of the origin. Let us set

$$\sigma_1^{\lambda,\mu}(x) = \lambda^{-3}\int_{\mathbb{R}^3} \alpha(\lambda^{-1}y)\frac{\chi(\mu[x-y])}{|x-y|^2}\,\mathrm{d}y = \alpha^\lambda \star \left(\frac{\chi^\mu}{|\cdot|^2}\right)(x) \quad \text{and} \quad \Sigma^{\lambda,\mu} = \sigma_1^{\lambda,\mu} \star \sigma_1^{\lambda,\mu},$$

where

$$\alpha^\lambda(x) = \lambda^{-3}\alpha(\lambda^{-1}x) \quad \text{and} \quad \chi^\mu(x) = \chi(\mu x).$$

Then each $\sigma_1^{\lambda,\mu}$ satisfies **(H2)**–**(H3)**. Moreover we can check that

$$\sigma_1^{\lambda,\mu}(x) = \mu^2\widetilde{\sigma_1}^{\lambda\mu}(\mu x), \qquad \Sigma^{\lambda,\mu}(x) = \mu\widetilde{\Sigma}^{\epsilon}(\mu x),$$

where

$$\widetilde{\sigma_1}^{\epsilon}(x) = \int \alpha^{\epsilon}(x-y)\frac{\chi(y)}{|y|^2}\,\mathrm{d}y, \qquad \widetilde{\Sigma}^{\epsilon} = \widetilde{\sigma_1}^{\epsilon} \star \widetilde{\sigma_1}^{\epsilon}.$$

Then Lemma 8.1 applies to this new sequence as well and Proposition 8.2 holds provided we can show that it converges to Σ^0 in the sense of (32). Such a form function appeared in [8]. The construction is based on the following two observations:

$$\frac{1}{|\cdot|^2} \star \frac{1}{|\cdot|^2}(x) = \frac{C}{|x|} = C\,\Sigma^0(x) \quad \text{where} \quad C = \int_{\mathbb{R}^3} \frac{\mathrm{d}y}{|y|^2|e_1-y|^2}$$

(e_1 being the first vector of the canonical basis), and

$$\Sigma^{\lambda,\mu} = (\alpha^\lambda \star \alpha^\lambda) \star \left(\frac{\chi^\mu}{|\cdot|^2} \star \frac{\chi^\mu}{|\cdot|^2} \right).$$

Then, at least formally, $\alpha^\lambda \star \alpha^\lambda \to (\int \alpha \star \alpha \, \mathrm{d}x)\delta_0$ when $\lambda \to 0$ and $(\chi^\mu/|\cdot|^2) \star (\chi^\mu/|\cdot|^2) \to (1/|\cdot|^2) \star (1/|\cdot|^2) = C\,\Sigma^0$ when $\mu \to 0$ and we can expect that $\Sigma^{\lambda,\mu}$ looks like Σ^0 when $\lambda, \mu \to 0$ provided $\int \alpha \, \mathrm{d}x = 1/\sqrt{C}$. The intuition is confirmed by the following claim.

Lemma 8.3 *If $\int \alpha \, \mathrm{d}x = 1/\sqrt{C}$, then the sequence $(\Sigma^{\lambda,\mu})_{\lambda,\mu>0}$ converges to Σ^0 in the sense of* (32) *when $(\lambda, \mu) \to (0, 0)$.*

This approach allows us to construct a large class of admissible form functions, not necessarily close de Σ^0 in the sense of (32), by using suitable rescalings that preserve the coercivity estimate as we did with the toy example 1. Indeed, for any α and χ defined as before, if the form function $\sigma_1 = \alpha \star (\chi/|\cdot|^2)$ is not in $\mathscr{A}$ we know, at least that up to rescaling α into $\alpha^\epsilon(x) = \epsilon^{-3}\alpha(\epsilon^{-1}x)$, that the form functions $\widetilde{\sigma_1}^\epsilon = \alpha^\epsilon \star (\chi/|\cdot|^2)$ belong to $\mathscr{A}$ provided ϵ is sufficiently small. With the previous notation the non empty mass interval I associated to the form function $\widetilde{\sigma_1}^\epsilon$ is given by $I = (\mu_0^{-1}M, \lambda_0\epsilon^{-1}M)$. It is also possible to rescale χ into $\chi^\epsilon(x) = \chi(\epsilon x)$ and obtain that form functions $\check{\sigma}_1^\epsilon = \alpha \star (\chi^\epsilon/|\cdot|^2)$ equally belong to $\mathscr{A}$ provided ϵ is sufficiently small (this second example uses the scaling relation $\sigma_1^{\lambda,\mu}(x) = \lambda^{-2}\check{\sigma}_1^{\lambda\mu}(\lambda^{-1}x)$). Moreover given an admissible function σ_1, we observe that $\sigma_1^{\lambda,\mu}(x) = \lambda\sigma_1(\mu x)$ is admissible too. We obtain this way form functions with arbitrary support size and L^∞_x-norm, which are non negative, non increasing, radially symmetric and concentrated around the origin. Such form functions are physically meaningful in the framework defined in [3]. Since they are simply derived by rescaling, we can check that the necessary coercivity estimate still holds, with constants that keep track of the rescaling, and they also provide stable ground states.

Proof of Lemma 8.3. Let $0 < R < \infty$ be fixed once for all. We decompose the difference $\Sigma^{\lambda,\mu} - \Sigma^0$ as follows

$$\begin{aligned}\Sigma^{\lambda,\mu}(x) - \Sigma^0(x) = &\left(\alpha^\lambda \star \alpha^\lambda\right) \star \left(\frac{\chi^\mu}{|\cdot|^2} \star \frac{\chi^\mu}{|\cdot|^2} - \frac{1}{|\cdot|^2} \star \frac{1}{|\cdot|^2} \right)(x) \\ &+ C \int \left(\alpha^\lambda \star \alpha^\lambda\right)(y) \left(\Sigma^0(x-y) - \Sigma^0(x)\right) \mathrm{d}y = I_1(x) + I_2(x).\end{aligned}$$

Bearing in mind that $\alpha^\lambda \star \alpha^\lambda(x) = \lambda^{-3}\alpha \star \alpha(\lambda^{-1}x)$, we readily obtain the convergence of $I_2\mathbf{1}_{|x|\le R}$ to 0 in the $L^{3/2}_x$-norm. Moreover, since the support of $\alpha^\lambda \star \alpha^\lambda$ shrinks to $\{0\}$ when $\lambda \to 0$ and since the function $x \mapsto 1/|x|$ is a Lipschitz function on every set of the form $\complement B(0, R)$ (with a Lipschitz constant $L(R)$ which blows up when $R \to 0$) we get

$$\|I_2\mathbf{1}_{|x|>R}\|_{L^\infty_x} \lesssim \mathrm{meas}\left(\mathrm{supp}\left(\alpha^\lambda \star \alpha^\lambda\right)\right) \underset{\lambda \to 0}{\longrightarrow} 0.$$

Next, for $y \in \operatorname{supp}(\alpha^\lambda \star \alpha^\lambda)$ with λ sufficiently small, $|x| > R$ implies $|x-y| > R/2$; it follows that

$$\begin{aligned}
&\|I_1 \mathbf{1}_{|x|>R}\|_{L^\infty_x} \\
&\quad \leq \left\| \left(\frac{\chi^\mu}{|\cdot|^2} \star \frac{\chi^\mu}{|\cdot|^2} - \frac{1}{|\cdot|^2} \star \frac{1}{|\cdot|^2} \right) \mathbf{1}_{|x|>R/2} \right\|_{L^\infty_x} = \sup_{|x|>R/2} \left| \int \frac{\chi^\mu(x-y)\chi^\mu(y) - 1}{|x-y|^2 |y|^2} \, \mathrm{d}y \right| \\
&\qquad\qquad \leq \sup_{|x|>R/2} \left| \int \frac{\chi^\mu(x-y)(\chi^\mu(y) - 1)}{|x-y|^2 |y|^2} \, \mathrm{d}y \right| + \sup_{|x|>R/2} \left| \int \frac{\chi^\mu(z) - 1}{|z|^2 |x+z|^2} \, \mathrm{d}z \right|.
\end{aligned}$$

Since $0 \leq \chi \leq 1$ and $\chi^\mu(x) = 1$ when $|x| \leq \mu^{-1}$ this estimate yields

$$\|I_1 \mathbf{1}_{|x|>R}\|_{L^\infty_x} \leq 4 \sup_{|x|>R/2} \int_{\complement B(0,\mu^{-1})} \frac{1}{|x-y|^2 |y|^2} \, \mathrm{d}y \underset{\mu \to 0}{\longrightarrow} 0.$$

It remains to prove that $I_1 \mathbf{1}_{|x|\leq R}$ converges to 0 in $L^{3/2}_x$-norm as $\lambda, \mu \to 0$. For $r \in (0, R)$ we split this quantity as follows

$$\|I_1 \mathbf{1}_{|x|\leq R}\|_{L^{3/2}_x} \leq \|I_1 \mathbf{1}_{|x|\geq r}\|_{L^{3/2}_x} + \|I_1 \mathbf{1}_{r<|x|\leq R}\|_{L^{3/2}_x}.$$

We have

$$\left| \left(\alpha^\lambda \star \alpha^\lambda\right) \star \left(\frac{\chi^\mu}{|\cdot|^2} \star \frac{\chi^\mu}{|\cdot|^2} - \frac{1}{|\cdot|^2} \star \frac{1}{|\cdot|^2} \right) \mathbf{1}_{|x|\leq r} \right| \leq 2C \left(\alpha^\lambda \star \alpha^\lambda\right) \star \Sigma^0 \mathbf{1}_{|x|\leq r}$$

and we have already seen that $C(\alpha^\lambda \star \alpha^\lambda) \star \Sigma^0 \mathbf{1}_{|x|\leq r}$ converges to $\Sigma^0 \mathbf{1}_{|x|\leq r}$ in the $L^{3/2}_x$-norm for any $0 < r < \infty$. Let $\eta > 0$. We can choose $r = r(\eta) > 0$ small enough and, next, find $\lambda(\eta)$ small enough so that for any $0 < \lambda < \lambda(\eta)$, we get

$$\|I_1 \mathbf{1}_{|x|\leq r}\|_{L^{3/2}_x} \leq 2\|(C(\alpha^\lambda \star \alpha^\lambda) \star \Sigma^0 - \Sigma^0)\mathbf{1}_{|x|\leq r}\|_{L^{3/2}_x} + 2\|\Sigma^0 \mathbf{1}_{|x|\leq r}\|_{L^{3/2}_x} \leq \eta.$$

Finally, the $L^{3/2}_x$-norm of $I_1 \mathbf{1}_{r<|x|\leq R}$ can be estimated as we did for the L^∞_x-norm of $I_1 \mathbf{1}_{|x|>R}$. Possibly at the price of taking $\lambda(\eta)$ smaller, if $|x| > r$ we have $|x-y| > r/2$ for any $y \in \operatorname{supp}(a^\lambda \star a^\lambda)$. It follows that

$$\|I_1 \mathbf{1}_{r<|x|\leq R}\|_{L^{3/2}_x} \leq \operatorname{meas}(B(0,R))^{2/3} \sup_{r/2<|x|\leq R} \int_{\complement B(0,\mu^{-1})} \frac{1}{|x-y|^2 |y|^2} \, \mathrm{d}y,$$

which can be made $\leq \eta$ for $0 < \mu < \mu(\eta)$, with $\mu(\eta)$ small enough. This ends the proof. ■

A Cauchy theory

From an energetic point of view, the natural functional spaces for the Cauchy theory of the Schrodinger-Wave equation are $C^0([0,T], H^1(\mathbb{R}^d_x))$ for the wave function u and

$$\mathcal{E}_T = C^0\left([0,T]; L^2\left(\mathbb{R}^d_x; \dot{H}^1(\mathbb{R}^n_z)\right)\right) \cap C^1\left([0,T]; L^2\left(\mathbb{R}^d_x; L^2(\mathbb{R}^n_z)\right)\right)$$

for the vibrational environment ψ. We are going to prove the global existence of solutions to (1a)–(1b), with Cauchy data (2), in these spaces, see Theorem 1.1. Throughout this appendix, we work, without loss of generality, with $c = 1$.

The proof of this theorem is quite classical: the most important part consists in applying Strichartz' estimates to the Schrödinger and the wave equation. In fact the main difficulty comes from the fact that Strichartz' estimates for (1a) lead to estimates of u in $L^q_t L^r_x$ norms whereas Strichartz' estimates for (1b) lead to estimates of ψ in $L^r_x L^q_t L^p_z$ norms. In order to combine these two estimates of different type, we need to permute Lebesgue-norms in time and space. For that purpose we will use Hölder and Young inequalities (and the fact that σ_1 and σ_2 are in any L^p space for $1 \leq p \leq +\infty$) in order to work with $L^q_t L^q_x$ norms.

Let us introduce some notation that we will use until the end of this section. First we denote by S the linear Schrödinger's group and by $(W, \dot{W})$ the free wave group: for any $u_0 \in L^2(\mathbb{R}^d_x)$, $S(t)u_0$ is the unique solution at time t of

$$\begin{cases} i\partial_t u + \Delta_x u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

and for any $(\psi_0, \psi_1) \in L^2(\mathbb{R}^d_x; \dot{H}^1(\mathbb{R}^n_z)) \times L^2(\mathbb{R}^d_x; L^2(\mathbb{R}^n_z))$, $\dot{W}(t)\psi_0 + W(t)\psi_1$ is the unique solution at time t of

$$\begin{cases} \partial^2_{tt}\psi - \Delta_z \psi = 0 \\ (\psi(0,x,z), \partial_t\psi(0,x,z)) = (\psi_0(x,z), \psi_1(x,z)) \end{cases}$$

With these notation we can now define (at least formally) the functions $\mathcal{L}$, $\mathcal{K}$ and Φ by

$$\begin{cases} \mathcal{L}(u,\psi) : t \longmapsto S(t)u_0 + \displaystyle\int_0^t S(t-s)\left[\left(\sigma_1 \star_x \int \sigma_2 \psi(s)\,\mathrm{d}z\right) u(s)\right]\,\mathrm{d}s \\ \mathcal{K}(u,\psi) : t \longmapsto \dot{W}(t)\psi_0 + W(t)\psi_1 + \displaystyle\int_0^t W(t-s)\left[-\sigma_2\sigma_1 \star_x |u(s)|^2\right]\,\mathrm{d}s \\ \Phi = (\mathcal{L}, \mathcal{K}) \end{cases}$$

where $u_0 \in H^1(\mathbb{R}^d_x)$ and $(\psi_0, \psi_1) \in L^2(\mathbb{R}^d_x; \dot{H}^1(\mathbb{R}^n_z)) \times L^2(\mathbb{R}^d_x; L^2(\mathbb{R}^n_z))$ are now fixed until the end of this section. From here it is obvious that any fixed point (u,ψ) of Φ defines a solution of (1a)–(1b) and (2). In order to apply the Banach-Picard fixed point theorem we have to specify on which space we define the function Φ. As already mentioned, since we wish to apply Strichartz estimates, we need that Φ is defined on a well adapted space for this approach. We introduce the following notations and spaces for that purpose. First let us define the Lebesgue exponent p_0 by

$$p_0 = \frac{2n}{n-2}. \tag{47}$$

Then, for any final time $T > 0$ we introduce the following Banach spaces: $X_T = L^\infty(0,T; H^1(\mathbb{R}^d_x))$, $Y_T = L^2(\mathbb{R}^d_x; L^\infty(0,T; L^{p_0}(\mathbb{R}^n_z)))$ and $Z_T = X_T \times Y_T$ endowed with the norm $\|u,\psi\|_{Z_T} = \|u\|_{X_T} + \|\psi\|_{Y_T}$.

We introduce these spaces because $(\infty, 2)$ is a *Schrödinger-admissible* pair and (∞, p_0) is a *wave-admissible* pair for $n \geq 3$. Let us briefly recall what are the definition of Schrödinger and wave-admissible pairs and what are Strichartz' estimates (we follow [16] and the interested reader can find further information about Strichartz' estimates in [11] and the references therein).

Definition A.1 *i) We say that the exponent pair* (q,r) *is* Schrödinger-admissible *if* $d \geq 1$, $q, r \geq 2$, $(q,r,d) \neq (2,\infty,2)$ *and*

$$\frac{1}{q} + \frac{d}{2r} = \frac{d}{4}.$$

ii) We say that the exponent pair (q,p) is wave-admissible *if $n \geq 2$, $q,p \geq 2$, $(q,p,n) \neq (2,\infty,3)$ and*

$$\frac{1}{q} + \frac{n-1}{2p} \leq \frac{n-1}{4}.$$

From now on for any exponent $a \geq 1$, a' will denote its conjugate exponent: $1/a + 1/a' = 1$.

Proposition A.2 (Strichartz estimates) *i) Let (q,r) and $(\bar{q},\bar{r})$ be Schrödinger-admissible pairs, $u_0 \in L^2(\mathbb{R}^d_x)$, $F \in L^{\bar{q}'}(0,T;L^{\bar{r}'}(\mathbb{R}^d_x))$ and let us denoted by u the unique solution of $i\partial_t u + \Delta_x u = F$ with initial data u_0. Then there exists a constant $C>0$ independent of T such that*

$$\|u\|_{L^q_t L^r_x} \leq C\left(\|u_0\|_{L^2_x} + \|F\|_{L^{\bar{q}'}_t L^{\bar{r}'}_x}\right) \tag{48}$$

ii) Let (q,p) and $(\bar{q},\bar{p})$ be wave-admissible pairs with $p,\bar{p} < +\infty$, $(\psi_0,\psi_1) \in \dot{H}^s(\mathbb{R}^n_z) \times \dot{H}^{s-1}(\mathbb{R}^n_z)$, $G \in L^{\bar{q}'}(0,T;L^{\bar{p}'}(\mathbb{R}^n_z))$ and let us denoted by ψ the unique solution of $\partial^2_{tt}\psi - \Delta_z\psi = G$ with initial data (ψ_0,ψ_1). Then, under the additional condition

$$\frac{1}{q} + \frac{n}{p} = \frac{n}{2} - s = \frac{1}{\bar{q}'} + \frac{n}{\bar{p}'} - 2, \tag{49}$$

there exists a constant $K>0$ independent of T such that

$$\|\psi\|_{L^q_t L^p_z} + \|\psi\|_{L^\infty_t \dot{H}^s_z} + \|\partial_t\psi\|_{L^\infty_t \dot{H}^{s-1}_z} \leq K\left(\|\psi_0\|_{\dot{H}^s_z} + \|\psi_1\|_{\dot{H}^{s-1}_z} + \|G\|_{L^{\bar{q}'}_t L^{\bar{p}'}_z}\right) \tag{50}$$

Remark A.3 *We will apply (50) with the Sobolev regularity $s=1$. With this regularity the exponent pairs $(q,p)=(\infty,p_0)$ and $(\infty,2)$ are wave-admissible and satisfies the additional condition (49).*

The following two Lemma justify that the application Φ is well defined on Z_T, sends Z_T into itself and admits a fixed point on it.

Lemma A.4 *There exists a constant $C>0$ independent of T such that*

$$\|\mathcal{L}(u,\psi)\|_{L^\infty_t L^2_x} \leq C\left(\|u_0\|_{L^2_x} + |T|\|\psi\|_{Y_T}\|u\|_{L^\infty_t L^2_x}\right), \tag{51a}$$

$$\|\nabla_x\mathcal{L}(u,\psi)\|_{L^\infty_t L^2_x} \leq C\left(\|\nabla_x u_0\|_{L^2_x} + |T|\|\psi\|_{Y_T}\left[\|u\|_{L^\infty_t L^2_x} + \|\nabla_x u\|_{L^\infty_t L^2_x}\right]\right), \tag{51b}$$

$$\begin{aligned}\|\mathcal{K}(u,\psi)\|_{Y_T} &+ \|\psi\|_{L^2_x L^\infty_t \dot{H}^1_z} + \|\partial_t\psi\|_{L^2_x L^\infty_t L^2_z} \\ &\leq C\left(\|\psi_0\|_{L^2_x \dot{H}^1_z} + \|\psi_1\|_{L^2_x L^2_z} + |T|\|u\|^2_{L^\infty_t L^2_x}\right),\end{aligned} \tag{51c}$$

and

$$\|\mathcal{L}(u,\psi) - \mathcal{L}(v,\varphi)\|_{L^\infty_t L^2_x} \leq C\,|T|\left(\|\psi\|_{Y_T}\|u-v\|_{L^\infty_t L^2_x} + \|\psi-\varphi\|_{Y_T}\|v\|_{L^\infty_t L^2_x}\right), \tag{52a}$$

$$\begin{aligned}\|\nabla_x(\mathcal{L}(u,\psi) - \mathcal{L}(v,\varphi))\|_{L^\infty_t L^2_x} \leq\ & C\,|T|(\|\psi\|_{Y_T}\left[\|u-v\|_{L^\infty_t L^2_x} + \|\nabla_x(u-v)\|_{L^\infty_t L^2_x}\right] \\ &+ \|\psi-\varphi\|_{Y_T}\left[\|v\|_{L^\infty_t L^2_x} + \|\nabla_x v\|_{L^\infty_t L^2_x}\right])\end{aligned} \tag{52b}$$

$$\|\mathcal{K}(u,\psi) - \mathcal{K}(v,\varphi)\|_{Y_T} \leq C|T|\left(\|u\|_{L^\infty_t L^2_x} + \|v\|_{L^\infty_t L^2_x}\right)\|u-v\|_{L^\infty_t L^2_x}. \tag{52c}$$

Lemma A.5 *There exists a universal constant $C_1 > 0$ such that for any final time $T > 0$ small enough, $\Phi : B_T \to B_T$, where*

$$B_T = \left\{ (u, \psi) \in Z_T \ : \ \|u, \psi\|_{Z_T} \leq C_1 (\|u_0\|_{H^1_x} + \|\psi_0\|_{L^2_z \dot{H}^1_z} + \|\psi_1\|_{L^2_x L^2_z}) \right\}.$$

Moreover, considering smaller T if necessary, Φ is indeed a contraction on B_T.

We postpone the proof of Lemma A.4 to the end of this section and we start by proving Lemma A.5 and Theorem 1.1.

Proof of Lemma A.5. We can summarize the estimates (51a)–(51c) as follows:

$$\|\Phi(u, \psi)\|_{Z_T} \leq C \left[\|u_0\|_{H^1_x} + \|\psi_0\|_{L^2_x \dot{H}^1_z} + \|\psi_1\|_{L^2_x L^2_z} + |T| \, \|u, \psi\|^2_{Z_T} \right].$$

Next, let $C_1 = 2C$; we thus obtain that for any $(u, \psi) \in B_T$,

$$\|\Phi(u, \psi)\|_{Z_T} \leq C \left[1 + C_1^2 \, |T| \, \left(\|u_0\|_{H^1_x} + \|\psi_0\|_{L^2_x \dot{H}^1_z} + \|\psi_1\|_{L^2_x L^2_z} \right) \right] \times \left(\|u_0\|_{H^1_x} + \|\psi_0\|_{L^2_x \dot{H}^1_z} + \|\psi_1\|_{L^2_x L^2_z} \right).$$

Since for T small enough,

$$C_1^2 \, |T| \, \left(\|u_0\|_{H^1_x} + \|\psi_0\|_{L^2_x \dot{H}^1_z} + \|\psi_1\|_{L^2_x L^2_z} \right) < 1,$$

we obtain that Φ sends B_T into B_T for T small enough. As previously, we can recast (52a)–(52c) as follows:

$$\|\Phi(u, \psi) - \Phi(v, \phi)\|_{Z_T} \leq C \, |T| \, (\|(u, \psi)\|_{Z_T} + \|v, \phi\|_{Z_T}) \, \|(u, \psi) - (v, \phi)\|_{Z_T}.$$

Therefore, for any $(u, \psi), (v, \phi) \in B_T$,

$$\|\Phi(u, \psi) - \Phi(v, \phi)\|_{Z_T} \leq 2C \, C_1 \left(\|u_0\|_{H^1_x} + \|\psi_0\|_{L^2_z \dot{H}^1_z} + \|\psi_1\|_{L^2_x L^2_z} \right) \, |T| \|(u, \psi) - (v, \phi)\|_{Z_T},$$

holds and Φ is a contraction as soon as T is small enough. ∎

Proof of Theorem 1.1. **Step 1: Local existence.** For T small enough Φ is a contraction on B_T, we thus know that (1a)–(1b) has a solution in Z_T. Then it is clear that for any solution $(u, \psi) \in Z_T$ of (1a)–(1b), $u \in L^\infty \left(0, T; H^1(\mathbb{R}^d_x)\right)$, $\psi \in L^2 \left(\mathbb{R}^d_x; L^\infty \left(0, T; \dot{H}^1(\mathbb{R}^n_z)\right)\right)$ and $\partial_t \psi \in L^2 \left(\mathbb{R}^d_x; L^\infty \left(0, T; L^2(\mathbb{R}^n_z)\right)\right)$ (for ψ its come from the Strichartz estimate (51c)). Moreover, using the fact that (u, ψ) is a fixed point of Φ and the expressions of $\mathcal{L}$ and $\mathcal{K}$ in terms of S and $(W, \dot{W})$, one can prove that indeed $u \in C^0 \left([0, T]; H^1(\mathbb{R}^d_x)\right)$, for almost every $x \in \mathbb{R}^d$, $(t, z) \mapsto \psi(t, x, z) \in C^0 \left([0, T]; \dot{H}^1(\mathbb{R}^n_z)\right)$ and $(t, z) \mapsto \partial_t \psi(t, x, z) \in C^0 \left([0, T]; L^2(\mathbb{R}^n_z)\right)$. We finish the proof by applying the following lemma (proved at the end of this section) to ψ and $\partial_t \psi$ in order to obtain that $\psi \in \mathcal{E}_T$.

Lemma A.6 *If $f \in L^2_x L^\infty_t$ and for almost every $x \in \mathbb{R}^d$, $t \mapsto f(t, x) \in C^0([0, T])$, then $f \in C^0 \left([0, T]; L^2(\mathbb{R}^d_x)\right)$.*

Step 2: Uniqueness. The uniqueness in B_T comes from the fixed point theorem and we can extend this uniqueness statement to the entire space Z_T. Then the uniqueness in $C^0_t H^1_x \times \mathcal{E}_T$ comes from the fact that any fixed point $(u,\psi) \in C_t^0 H_x^1 \times \mathcal{E}_T$ of Φ is also an element of Z_T (thanks to the estimate (51c), we get that ψ is in Y_T).

Step 3: Global existence. Since the time T in Lemma A.5 depends only on universal constants and on

$$\|u_0\|_{H_x^1} + \|\psi_0\|_{L_x^2 \dot{H}_z^1} + \|\psi_1\|_{L_x^2 L_z^2},$$

the first two steps of this proof allow us to obtain the following proposition.

Proposition A.7 *Let $n \geq 3$. Then for any $u_0 \in H^1(\mathbb{R}_x^d)$ and $(\psi_0,\psi_1) \in L^2(\mathbb{R}_x^d; \dot{H}^1(\mathbb{R}_z^n)) \times L^2(\mathbb{R}_x^d; L^2(\mathbb{R}_z^n))$, there exists $T^\star > 0$ such that for any $0 < T < T^\star$, the problem* (1a)–(1b) *and* (2) *admits a unique solution $(u,\psi) \in C^0\left([0,T]; H^1(\mathbb{R}_x^d)\right) \times \mathcal{E}_T$ on $[0,T]$. Moreover, if for some $0 < T \leq T^\star$,*

$$\limsup_{t \nearrow T} \|u(t)\|_{H_x^1} + \|\psi(t)\|_{L_x^2 \dot{H}_z^1} + \|\partial_t \psi(t)\|_{L_x^2 L_z^2} < +\infty,$$

then, actually, $T < T^\star$.

Then in order to obtain the global existence we have to justify that the quantity

$$\|u(t)\|_{H_x^1} + \|\psi(t)\|_{L_x^2 \dot{H}_z^1} + \|\partial_t \psi(t)\|_{L_x^2 L_z^2}$$

does not blow up in finite time. Thanks to the mass conservation of the wave function u ($M = \|u(t)\|_{L_x^2}$ is constant in time) and thanks to (51c) we get

$$\|u(t)\|_{H_x^1} + \|\psi(t)\|_{L_x^2 \dot{H}_z^1} + \|\partial_t \psi(t)\|_{L_x^2 L_z^2} \lesssim M + \|\nabla_x u(t)\|_{L_x^2} + \|\psi_0\|_{L_x^2 \dot{H}_z^1} + \|\psi_1\|_{L_x^2 L_z^2} + |t| M,$$

and it only remains to control $\|\nabla_x u(t)\|_{L_x^2}$. For that purpose we use the energy conservation (14) in order to obtain

$$\frac{1}{2}\|\nabla_x u(t)\|_{L_x^2} + \int \left(\sigma_1 \star \int \sigma_2 \psi(t)\,\mathrm{d}z\right) |u(t)|^2\,\mathrm{d}x \leq \mathcal{E}_{\mathrm{Schr}}(t) = \mathcal{E}_{\mathrm{Schr}}(0).$$

Then if $\|\nabla_x u(t)\|_{L_x^2}$ blows up in finite time, $|\int (\sigma_1 \star \int \sigma_2 \psi(t)\,\mathrm{d}z)|u(t)|^2\,\mathrm{d}x|$ has to blows up in finite time too. But

$$\begin{aligned}
\left\| \int (\sigma_1 \star \int \sigma_2 \psi\,\mathrm{d}z)|u|^2\,\mathrm{d}x \right\|_{L_t^\infty} &\leq M^2 \left\| \sigma_1 \star \int \sigma_2 \psi\,\mathrm{d}z \right\|_{L_t^\infty L_x^\infty} \\
&= M^2 \left\| \sigma_1 \star \int \sigma_2 \psi\,\mathrm{d}z \right\|_{L_x^\infty L_t^\infty} \leq M^2 \|\sigma_2\|_{L_z^{p_0'}} \left\| \sigma_1 \star \|\psi\|_{L_z^{p_0}} \right\|_{L_x^\infty L_t^\infty} \\
&\leq M^2 \|\sigma_2\|_{L_z^{p_0'}} \left\| \sigma_1 \star \|\psi\|_{L_t^\infty L_z^{p_0}} \right\|_{L_x^\infty} \leq M^2 \|\sigma_2\|_{L_z^{p_0'}} \|\sigma_1\|_{L_x^2} \|\psi\|_{L_x^2 L_t^\infty L_z^{p_0}}, \quad (53)
\end{aligned}$$

and eventually estimate (51c) tells us that $|\int (\sigma_1 \star \int \sigma_2 \psi(t)\,\mathrm{d}z)|u(t)|^2\,\mathrm{d}x|$ grows at most linearly in time. ∎

Remark A.8 *In fact the proof of the global existence gives us the additional information that the quantities $\|\nabla_x u(t)\|_{L_x^2}$, $\|\psi(t)\|_{L_x^2 \dot{H}_z^1} + \|\partial_t \psi(t)\|_{L_x^2 L_z^2}$ and $|\int (\sigma_1 \star \int \sigma_2 \psi(t)\,\mathrm{d}z)|u(t)|^2\,\mathrm{d}x|$ grow at most linearly in time.*

We finish this section with the proofs of Lemma A.4 and Lemma A.6.

Proof of Lemma A.4. *Estimate* (51a). We apply apply the Strichartz estimate (48) to $\mathcal{L}(u,\psi)$ with the Schrödinger-admissible pair $(\infty,2)$ on both side to obtain

$$\|\mathcal{L}(u,\psi)\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2_x} + \left\| \left(\sigma_1 \star_x \int \sigma_2 \psi \,\mathrm{d}z\right) u\right\|_{L^1_t L^2_x}.$$

Then, thanks to the following estimate

$$\begin{aligned}\left\| \left(\sigma_1 \star_x \int \sigma_2 \psi \,\mathrm{d}z\right) u\right\|_{L^1_t L^2_x} &\leq |T| \left\| \left(\sigma_1 \star_x \int \sigma_2 \psi \,\mathrm{d}z\right) u\right\|_{L^\infty_t L^2_x} \\ &\leq \left\| \sigma_1 \star_x \int \sigma_2 \psi \,\mathrm{d}z\right\|_{L^\infty_t L^\infty_x} \|u\|_{L^\infty_t L^2_x},\end{aligned}$$

and thanks to (53), we eventually obtain

$$\|\mathcal{L}(u,\psi)\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2_x} + |T| \, \|\psi\|_{Y_T} \, \|u\|_{L^\infty_t L^2_x}.$$

Estimate (51b). Since

$$\begin{aligned}\nabla_x \mathcal{L}(u,\psi)(t) = {} & S(t)\nabla_x u_0 \\ & + \int_0^t S(t-s)\left[\left(\nabla_x\sigma_1 \star \int \sigma_2\psi(s)\,\mathrm{d}z\right)u(s) + \left(\sigma_1 \star \int \sigma_2 \psi(s)\,\mathrm{d}z\right)\nabla_x u(s)\right]\mathrm{d}s,\end{aligned}$$

we just apply the same estimates as before.

Estimate (51c). We apply for almost every $x \in \mathbb{R}^d$ the Strichartz estimate (50) to $\mathcal{K}(u,\psi)(x)$ with the wave-admissible pair (∞, p_0) on the left hand side and $(\infty,2)$ on the right hand side

$$\begin{aligned}\|\mathcal{K}(u,\psi)(x)\|_{L^\infty_t L^{p_0}_z} + \|\psi(x)\|_{L^\infty_t \dot{H}^1_z} + \|\partial_t \psi(x)\|_{L^\infty_t L^2_z} & \\ \lesssim \|\psi_0(x)\|_{\dot{H}^1_z} + \|\psi_1(x)\|_{L^2_z} + \left\|\sigma_2\,\sigma_1 \star |u|^2(x)\right\|_{L^1_t L^2_z}&.\end{aligned}$$

Then, since

$$\left\|\sigma_2\,\sigma_1 \star |u|^2(x)\right\|_{L^1_t L^2_z} = \|\sigma_2\|_{L^2_z} \|\sigma_1 \star |u|^2(x)\|_{L^1_t} \leq \|\sigma_2\|_{L^2_z} |\sigma_1| \star \|u\|^2_{L^2_t}(x)$$

we can pass in L^2_x-norm to obtain

$$\left\|\sigma_2\,\sigma_1 \star |u|^2\right\|_{L^2_x L^1_t L^2_z} \leq \|\sigma_2\|_{L^2_z} \left\| |\sigma_1| \star \|u\|^2_{L^2_t}\right\|_{L^2_x}.$$

Here, thanks to the Young inequality we have

$$\left\| |\sigma_1| \star \|u\|^2_{L^2_t}\right\|_{L^2_x} \leq \|\sigma_1\|_{L^2_x} \left\| \|u\|^2_{L^2_t}\right\|_{L^1_x} = \|\sigma_1\|_{L^2_x} \|u\|^2_{L^2_t L^2_x} \leq \|\sigma_1\|_{L^2_x} |T| \, \|u\|^2_{L^\infty_t L^2_x},$$

and we eventually obtain

$$\|\mathcal{K}(u,\psi)\|_{L^2_x L^\infty_t L^{p_0}_z} + \|\psi\|_{L^2_x L^\infty_t \dot{H}^1_z} + \|\partial_t\psi\|_{L^2_x L^\infty_t L^2_z} \lesssim \|\psi_0\|_{L^2_x \dot{H}^1_z} + \|\psi_1\|_{L^2_x L^2_z} + |T| \, \|u\|^2_{L^\infty_t L^2_x}.$$

Estimates (52a)**,** (52b) **and** (52c)**.** Since

$$\begin{aligned}&\mathcal{L}(u,\psi)(t) - \mathcal{L}(v,\varphi)(t) = \\ &\quad \int_0^t S(t-s)\left[\left(\sigma_1 \star_x \int \sigma_2\psi(s)\,\mathrm{d}z\right)(u(s)-v(s)) + \left(\sigma_1 \star_x \int \sigma_2(\psi(s)-\varphi(s))\,\mathrm{d}z\right)v(s)\right]\mathrm{d}s\end{aligned}$$

and

$$\mathcal{K}(u,\psi)(t)-\mathcal{K}(v,\varphi)(t)=\\ \int_0^t W(t-s)\left[-\sigma_2\sigma_1\star_x\left([u(s)-v(s)]\bar{u}(s)+v(s)[\bar{u}(s)-\bar{v}(s)]\right)\right]\mathrm{d}s,$$

we just follow closely the proof of (51a), (51b) and (51c). ∎

Proof of Lemma A.6. Let us fix $\varepsilon>0$ and $t\in[0,T]$. We know that for all $x\in\mathbb{R}^d$ and for all $\eta>0$, there exists $\delta(\eta,t,x)\geq 0$ such that if $|t-s|\leq\delta(\eta,t,x)$, then $|f(t,x)-f(s,x)|\leq\eta$. Note that in fact $\delta(\eta,t,x)$ is positive for almost every $x\in\mathbb{R}^d$. Moreover, since $f\in L^2_xL^\infty_t$ we now that

$$\int_{\mathbb{R}^d}\mathbf{1}_{|x|\geq R}\|f(x)\|^2_{L^\infty_t}\,\mathrm{d}x\underset{R\to\infty}{\longrightarrow}0.$$

Let $\delta>0$. Let us also introduce the following subset of $\mathbb{R}^d_x$

$$B^{R,\eta}_{t,\delta}=\left\{x\in\mathbb{R}^d\text{ such that }|x|\leq R\text{ and }\delta(\eta,t,x)\leq\delta\right\}.$$

Note that $\mathrm{meas}(B^{R,\eta}_{t,\delta})\to 0$ when $\delta\to 0$. Then for all $R,\eta,\delta>0$ and for all s such that $|t-s|\leq\delta$,

$$\begin{aligned}\|f(t)-f(s)\|_{L^2_x}&\leq\|\mathbf{1}_{|x|\geq R}(f(t)-f(s))\|_{L^2_x}+\|\mathbf{1}_{|x|\leq R}(f(t)-f(s))\|_{L^2_x}\\&\leq 2\,\|\mathbf{1}_{|x|\geq R}f\|_{L^2_xL^\infty_t}+\eta\,\mathrm{meas}\,(B(0,R))^{1/2}+2\,\mathrm{meas}\left(B^{R,\eta}_{t,\delta}\right)\|f\|_{L^2_xL^\infty_t}.\end{aligned}$$

We can pick R large enough to obtain

$$2\,\|\mathbf{1}_{|x|\geq R}f\|_{L^2_xL^\infty_t}\leq\frac{\varepsilon}{3},$$

then we fix η small enough to get

$$\eta\,\mathrm{meas}\,(B(0,R))^{1/2}\leq\frac{\varepsilon}{3},$$

and we eventually fix δ small enough to get

$$2\,\mathrm{meas}\left(B^{R,\eta}_{t,\delta}\right)\|f\|_{L^2_xL^\infty_t}\leq\frac{\varepsilon}{3}.$$

∎

B Semi-classical analysis

In this section we rescale the Schrödinger-Wave system as follows

$$ih\,\partial_t u_h+\frac{h^2}{2}\Delta_x u_h=\left(\sigma_1\star_x\int\sigma_2\psi_h(t)\,\mathrm{d}z\right)u_h,\qquad t\in\mathbb{R},\ x\in\mathbb{R}^d \tag{54a}$$

$$\partial_t\psi_h=\chi_h,\qquad t\in\mathbb{R},\ x\in\mathbb{R}^d,\ z\in\mathbb{R}^n \tag{54b}$$

$$\partial_t\chi_h=c^2\Delta_z\psi_h-c^2\sigma_2(z)\left(\sigma_1\star_x|u_h(t)|^2\right)(x),\qquad t\in\mathbb{R},\ x\in\mathbb{R}^d,\ z\in\mathbb{R}^n \tag{54c}$$

where $h>0$ denotes (a dimensionless version of) the Planck constant. We wish to investigate the behavior of this rescaled system when $h\to 0$. This is expected to establish a connection between the classical and quantum models, see [26]. More precisely for every $h>0$ we consider the Wigner transform of u_h

$$W_h(t,x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot y} u_h(t,x+\frac{h}{2}y)\,\bar{u}_h(t,x-\frac{h}{2}y)\,\mathrm{d}y$$

and we address the question of the asymptotic behavior of (W_h,ψ_h,χ_h) when h goes to 0. Our goal is to prove that (W_h,ψ_h,χ_h) admits a limit and this limit is a solution of the Vlasov-Wave system (6a)–(6b). For that purpose let us introduce some notations and assumptions.

We consider a sequence of initial data $(u_0^h)_{h>0}\subset H_x^1$, $(\psi_0^h)_{h>0}\subset L_x^2\dot{H}_z^1$ and $(\chi_0^h)_{h>0}\subset L_x^2L_z^2$ such that

(H5) the quantities $\|u_h\|_{L_x^2}$ and

$$\mathscr{E}_{0,+}^h = \frac{h^2}{2}\int_{\mathbb{R}^d}|\nabla_x u_0^h|^2\,\mathrm{d}x + \int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\psi_0^h\,\mathrm{d}z\right)_+|u_0^h|^2\,\mathrm{d}x \\ + \frac{1}{2c^2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}|\chi_0^h|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\psi_0^h|^2\,\mathrm{d}x\,\mathrm{d}z$$

are uniformly bounded with respect to h.

Remark B.1 *i) Assumption **(H5)** guarantees us that the sequences (ψ_0^h) and (χ_0^h) are uniformly bounded with respect to h respectively in $L_x^2\dot{H}_z^1$ and $L_x^2L_z^2$. Hence, there exists $\psi_0\in L_x^2\dot{H}_z^1$ and $\chi_0\in L_x^2L_z^2$ such that, sub-sequencse still labelled $(\psi_0^h)_{h>0}$ and $(\chi_0^h)_{h>0}$ converge respectively to ψ_0 in $L_x^2\dot{H}_z^1-$weakly and χ_0 in $L_x^2L_z^2-$weakly.*
ii) Moreover, since the rescaled Hamiltonian

$$\mathscr{E}^h(t) = \frac{h^2}{2}\int_{\mathbb{R}^d}|\nabla_x u_h(t)|^2\,\mathrm{d}x + \int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\psi_h(t)\,\mathrm{d}z\right)|u_h(t)|^2\,\mathrm{d}x \\ + \frac{1}{2c^2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}|\chi_h(t)|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\psi_h(t)|^2\,\mathrm{d}x\,\mathrm{d}z$$

is conserved by the system (54a)–(54c)*, we have*

$$\begin{aligned} 0 \le \frac{h^2}{2}\int_{\mathbb{R}^d}|\nabla_x u_h(t)|^2\,\mathrm{d}x + \frac{1}{2c^2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}|\chi_h(t)|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\psi_h(t)|^2\,\mathrm{d}x\,\mathrm{d}z \\ = \mathscr{E}^h(0) - \int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\psi_h(t)\,\mathrm{d}z\right)|u_h(t)|^2\,\mathrm{d}x \\ \le \mathscr{E}_{0,+}^h - \int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\psi_h(t)\,\mathrm{d}z\right)|u_h(t)|^2\,\mathrm{d}x. \end{aligned}$$

Then thanks to (53) *coupled with the mass conservation of the wave function u_h and* (51c) *we have*

$$\left\|\int_{\mathbb{R}^d}\left(\sigma_1\star\int_{\mathbb{R}^n}\sigma_2\psi_h(t)\,\mathrm{d}z\right)|u_h(t)|^2\,\mathrm{d}x\right\|_{L_t^\infty} \lesssim \left(\|\psi_0^h\|_{L_x^2\dot{H}_z^1} + \|\chi_0^h\|_{L_x^2L_z^2} + |T|\|u_0^h\|_{L_x^2}\right)\|u_0^h\|_{L_x^2}^2,$$

that means $h^2\|\nabla_x u_h(t)\|_{L_x^2}^2$, $\|\chi_h(t)\|_{L_x^2L_z^2}$ and $\|\psi_h(t)\|_{L_x^2\dot{H}_z^1}$ are uniformly bounded with respect to h and $t\in[0,T]$.

One can easily check that the Wigner transform W_h associated to a solution u_h of (54a) satisfies the following equation

$$\partial_t W_h + \xi \cdot \nabla_x W_h + K_h \star_\xi W_h = 0, \tag{55}$$

where

$$K_h(t,x,\xi) = \frac{i}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi\cdot y} \frac{1}{h} \left(\Phi_h(t, x + \frac{h}{2}y) - \Phi_h(t, x - \frac{h}{2}y) \right) \mathrm{d}y. \tag{56}$$

This follows by direct inspection when u_h is a strong solution of (54a), which is the case if u_0^h is regular enough; dealing with weak solutions requires a step by regularization and approximation.

According to [26], we introduce the separable Banach space

$$\mathcal{A} = \left\{ \varphi \in C^0(\mathbb{R}^d_x \times \mathbb{R}^d_\xi) \ \text{ s.t. } \ \mathcal{F}_\xi \varphi(x,y) \in L^1 \left(\mathbb{R}^d_y; C^0(\mathbb{R}^d_x) \right) \right\}$$

equipped with the norm

$$\|\varphi\|_{\mathcal{A}} = \|\mathcal{F}_\xi \varphi\|_{L^1_y C^0_x} = \int_{\mathbb{R}^d} \sup_x |\mathcal{F}_\xi \varphi(x,y)| \, \mathrm{d}y,$$

and notice that the space

$$\mathcal{B} = \left\{ \varphi \in \mathcal{S} \ \text{ s.t. } \ \mathcal{F}_\xi \varphi \in C^\infty_c(\mathbb{R}^d_x \times \mathbb{R}^d_y) \right\}$$

is dense in $\mathcal{A}$. We also denote by $\mathcal{M} = \mathcal{M}(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ the space of bounded measures on $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$, and $\mathcal{M}_+$ its positive cone.

Theorem B.2 *Let **(H1)**–**(H2)** and **(H5)** be fulfilled. Up to a sub-sequence, the families $(W_h)_{h>0}$, $(\psi_h)_{h>0}$ and $(\chi_h)_{h>0}$ converge respectively to $\mu \in C^0([0,T]; \mathcal{M} - w\star)$, $\psi \in C^0([0,T]; L^2_x \dot{H}^1_z - w)$ and $\chi \in C^0([0,T]; L^2_x L^2_z - w)$ respectively in the spaces $C^0([0,T]; \mathcal{A}' - w\star)$, $C^0([0,T]; L^2_x \dot{H}^1_z - w)$ and $C^0([0,T]; L^2_x L^2_z - w)$. Moreover (μ, ψ, χ) is a solution of the Vlasov-Wave system*

$$\begin{aligned}
&\partial_t \mu + \mathrm{div}_x(\xi\mu) - \mathrm{div}_\xi \left(\nabla_x \left[\sigma_1 \star_x \int \sigma_2 \psi(t) \, \mathrm{d}z \right] \mu \right) = 0, && \textit{in } \mathcal{D}'\left((0,T); \mathcal{B}'\right), \\
&\partial_t \psi = \chi, && \textit{in } \mathcal{D}'\left((0,T) \times \mathbb{R}^d_x \times \mathbb{R}^n_z\right), \\
&\partial_t \chi = c^2 \Delta_z \psi - \sigma_2(z) \left(\sigma_1 \star_x \int \mathrm{d}\mu(\xi) \right)(x), && \textit{in } \mathcal{D}'\left((0,T) \times \mathbb{R}^d_x \times \mathbb{R}^n_z\right).
\end{aligned}$$

The proof follows closely the analysis of [26]; the main difference being that here we have to control also what happens as $h \to 0$ for the wave part of the system (54a)–(54c). Note that if the sequence of initial data is supposed to converge, then, by uniqueness of the solution of the limit equation [8, Theorem 4], the entire sequence $(W_h, \psi_h, \chi_h)_{h>0}$ converges.

Proof. **Step 1: Convergence of $(\psi_h)_{h>0}$.** Thanks to Remark B.1 we already know that the sequence $(\psi_h)_{h>0}$ is bounded in $L^\infty(0,T; L^2_x \dot{H}^1_z)$. Since any closed ball of $L^2_x \dot{H}^1_z$ is metrizable and compact for the weak topology, we are going to apply the Ascoli-Arzela theorem in order to justify that $(\psi_h)_{h>0}$ admits a converging sub-sequence in $C^0_t(L^2_x \dot{H}^1_z - w)$. For that purpose it only remains to show that $(\psi_h)_{h>0}$ is equi-continuous in $C^0_t(L^2_x \dot{H}^1_z - w)$. In fact, it is sufficient to prove that the family $\{t \mapsto \langle \psi_h(t), g\rangle_{L^2_x \dot{H}^1_z\}}$ is equi-continuous for every g in a dense countable subset of $L^2_x \dot{H}^1_z$. Details on this argument can be found e. g. in [25, Appendix C]. For any $g \in C_c^\infty(\mathbb{R}^d_x \times \mathbb{R}^n_z)$,

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi_h(t), g\rangle_{L^2_x \dot{H}^1_z} \right| = \left| \iint_{\mathbb{R}^d \times \mathbb{R}^n} \hat{\chi}_h(t,k,\zeta)|\zeta|^2 \overline{\hat{g}(k,\zeta)}\, \mathrm{d}k\, \mathrm{d}\zeta \right| \leq \|\chi_h(t)\|_{L^2_x L^2_z} \|g\|_{L^2_x H^2_z}$$

is uniformly bounded in h and $t \in [0,T]$ (see Remark B.1) and the Ascoli-Arzela theorem insures us that, up to a sub-sequence, $(\psi_h)_{h>0}$ converges in $C^0([0,T]; L^2_x \dot{H}^1_z - w)$ to $\psi \in C^0([0,T]; L^2_x \dot{H}^1_z - w)$.

Step 2: Convergence of $(\chi_h)_{h>0}$. As in the previous step Remark B.1 insures us that the sequence $(\chi_h)_{h>0}$ is bounded in $L^\infty(0,T; L^2_x L^2_z)$. Moreover, for any $g \in C_c^\infty(\mathbb{R}^d_x \times \mathbb{R}^n_z)$,

$$\begin{aligned} &\left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \chi_h(t), g\rangle_{L^2_x L^2_z} \right| \\ &\qquad \leq c^2 \left| \iint_{\mathbb{R}^d \times \mathbb{R}^n} \nabla_z \psi_h(t) \cdot \nabla_z g \, \mathrm{d}x\, \mathrm{d}z \right| + c^2 \left| \iint_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_2(z) \sigma_1 \star |u_h(t)|^2(x)\, g(x,z)\, \mathrm{d}x\, \mathrm{d}z \right| \\ &\qquad\qquad \leq \|\psi_h\|_{L^2_x \dot{H}^1_z} \|g\|_{L^2_x H^1_z} + \|\sigma_1\|_{L^2_x} \|\sigma_2\|_{L^2_z} \|u_h(t)\|^2_{L^2_x} \|g\|_{L^2_x L^2_z} \end{aligned}$$

is uniformly bounded in h and $t \in [0,T]$ (see Remark B.1). Eventually the Ascoli-Arzela theorem insures us that, up to a sub-sequence, (χ_h) converges in $C^0([0,T]; L^2_x L^2_z - w)$ to $\chi \in C^0([0,T]; L^2_x L^2_z - w)$.

Step 3: Equation on ψ. Since χ_h converges to χ in $C^0([0,T]; L^2_x L^2_z - w)$ we obtain directly that for any $g \in C_c^\infty(\mathbb{R}^d_x \times \mathbb{R}^n_z)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi_h(t), g\rangle_{\mathcal{D}',\mathcal{D}} = \iint_{\mathbb{R}^d \times \mathbb{R}^n} \chi_h(t)\, g \, \mathrm{d}x\, \mathrm{d}z \underset{h \to 0}{\longrightarrow} \langle \chi(t), g\rangle_{\mathcal{D}',\mathcal{D}}$$

the convergence being uniform on $[0,T]$. Note that here, since the duality product on $L^2_x \dot{H}^1_z$ is not compatible with the duality product in $\mathcal{D}'$, we have to say something in order to justify the following convergence

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi_h(t), g\rangle_{\mathcal{D}',\mathcal{D}} \underset{h \to 0}{\longrightarrow} \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi(t), g\rangle_{\mathcal{D}',\mathcal{D}} \quad \text{in } \mathcal{D}'(0,T).$$

Since for any $f \in C_c^\infty(0,T)$,

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi_h, g\rangle_{\mathcal{D}'}, f \right\rangle_{\mathcal{D}'(0,T)} = -\int_0^T \langle \psi_h(t), g\rangle_{\mathcal{D}'} f'(t)\, \mathrm{d}t$$

we have to justify the uniform convergence in time of $\langle \psi_h(t), g\rangle_{\mathcal{D}'}$ to $\langle \psi(t), g\rangle_{\mathcal{D}'}$. For any $g \in C_c^\infty(\mathbb{R}^d_x \times \mathbb{R}^n_z)$, we have

$$\langle \psi_h(t), g\rangle_{\mathcal{D}'} = \iint_{\mathbb{R}^d \times \mathbb{R}^n} |\zeta| \hat{\psi}_h(t,k,\zeta)\, |\zeta| \frac{\overline{\hat{g}(k,\zeta)}}{|\zeta|^2} \, \mathrm{d}k\, \mathrm{d}\zeta.$$

The condition $n \geq 3$ implies that $\mathcal{F}^{-1}(\hat{g}(k,\zeta)/|\zeta|^2)$ lies in $L^2_x \dot{H}^1_z$, and the convergence of ψ_h to ψ in $C^0([0,T]; L^2_x \dot{H}^1_z - w)$ allows us to conclude. Eventually we have proved that $\partial_t \psi = \chi$ in $\mathcal{D}'$.

Step 4: Equation on χ. Let us temporarily assume that $|u_h(t)|^2$ converges to a certain $\rho \in C^0([0,T]; \mathcal{M} - w\star)$ (see **Step 7**). For any $g \in C_c^\infty(\mathbb{R}^d_x \times \mathbb{R}^n_z)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \chi_h(t), g\rangle_{\mathcal{D}',\mathcal{D}} = -c^2 \iint_{\mathbb{R}^d \times \mathbb{R}^n} \nabla_z \psi_h(t) \cdot \nabla_z g \, \mathrm{d}x\, \mathrm{d}z - c^2 \iint_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_2\, \sigma_1 \star |u_h(t)|^2\, g \, \mathrm{d}x\, \mathrm{d}z \tag{58}$$

The weak convergence of $(\psi_h)_{h>0}$ insures us that

$$-c^2 \iint_{\mathbb{R}^d\times\mathbb{R}^n} \nabla_z \psi_h(t)\cdot\nabla_z g \,\mathrm{d}x\,\mathrm{d}z \underset{h\to 0}{\longrightarrow} -c^2 \iint_{\mathbb{R}^d\times\mathbb{R}^n} \nabla_z \psi(t)\cdot\nabla_z g \,\mathrm{d}x\,\mathrm{d}z$$

and, if we rewrite the second term of the right hand side of (58) as follows

$$c^2 \iint_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_2\,\sigma_1 \star |u_h(t)|^2\, g \,\mathrm{d}x\,\mathrm{d}z = c^2 \int_{\mathbb{R}^d} |u_h(t,y)|^2 \left(\int_{\mathbb{R}^n} \sigma_2\,\sigma_1\star g(y)\,\mathrm{d}z\right)\,\mathrm{d}y,$$

the weak convergence of $|u_h|^2$ leads to

$$c^2 \iint_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_2\,\sigma_1 \star |u_h(t)|^2\, g \,\mathrm{d}x\,\mathrm{d}z \underset{h\to 0}{\longrightarrow} c^2 \iint_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_2\,\sigma_1\star\rho(t)\, g\,\mathrm{d}x\,\mathrm{d}z.$$

These two convergences hold uniformly in time and we eventually obtain

$$\partial_t\chi = c^2\Delta_z\psi - c^2\sigma_2\,\sigma_1\star\rho(t) \quad \text{in } \mathcal{D}'\left((0,T)\times\mathbb{R}^d_x\times\mathbb{R}^n_z\right).$$

Step 5: Convergence of $(W_h)_{h>0}$. We first prove that the sequence $(W_h)_{h>0}$ is bounded in $L^\infty(0,T;\mathcal{A}')$. Since

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} W_h(t,x,\xi)\varphi(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi = \frac{1}{(2\pi)^d}\iint_{\mathbb{R}^d\times\mathbb{R}^d} u_h(t,x+\frac{h}{2}y)\bar{u}_h(t,x-\frac{h}{2}y)\mathcal{F}_\xi\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

we obtain directly

$$\begin{aligned}\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} W_h(t,x,\xi)\varphi(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\right| \\ &\le \frac{1}{(2\pi)^d}\left(\sup_y \int_{\mathbb{R}^d}\left|u_h(t,x+\frac{h}{2}y)\bar{u}_h(t,x-\frac{h}{2}y)\right|\,\mathrm{d}x\right)\left(\sup_x\int_{\mathbb{R}^d}|\mathcal{F}_\xi\varphi(x,y)|\,\mathrm{d}y\right) \\ &\le \frac{1}{(2\pi)^d}\|u_h(t)\|^2_{L^2_x}\|\varphi\|_{\mathcal{A}},\end{aligned}$$

which insures us

$$\|W_h(t)\|_{\mathcal{A}'} \le \frac{1}{(2\pi)^d}\|u_h(t)\|^2_{L^2_x}$$

is bounded with respect to h and t. Since any closed ball of $\mathcal{A}'$ is metrizable and compact for the weak-$\star$ topology, we will apply again the Ascoli-Arzela theorem in order to justify that $(W_h)_{h>0}$ admits a converging sub-sequence in $C^0_t(\mathcal{A}'-w\star)$. For that purpose we will prove that for any $\varphi\in\mathcal{B}$, the functions $t\mapsto\langle W_h(t),\varphi\rangle_{\mathcal{A}',\mathcal{A}}$ are equi-continuous. Direct computations yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle W_h(t),\varphi\rangle_{\mathcal{A}',\mathcal{A}} &= -\iint_{\mathbb{R}^d\times\mathbb{R}^d} W_h(t,x,\xi)\,\xi\cdot\nabla_x\varphi(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\quad + \iint_{\mathbb{R}^d\times\mathbb{R}^d} W_h(t,x,\eta)\left(\int_{\mathbb{R}^d} K_h(t,x,\xi-\eta)\varphi(x,\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}x\,\mathrm{d}\eta, \quad (59)\end{aligned}$$

with

$$\begin{aligned}L_h(t,x,\eta) &:= \int_{\mathbb{R}^d} K_h(t,x,\xi-\eta)\varphi(x,\xi)\,\mathrm{d}\xi \\ &= \frac{i}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\eta\cdot y}\frac{1}{h}\left(\Phi_h(t,x+\frac{h}{2}y)-\Phi_h(t,x-\frac{h}{2}y)\right)\mathcal{F}_\xi\varphi(x,y)\,\mathrm{d}y\end{aligned}$$

and

$$\mathcal{F}_\eta L_h(t,x,y) = \frac{i}{h}\left(\Phi_h(t,x+\frac{h}{2}y) - \Phi_h(t,x-\frac{h}{2}y)\right)\mathcal{F}_\xi\varphi(x,y).$$

From (59) we get for any $\varphi \in \mathcal{B}$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\langle W_h(t),\varphi\right\rangle_{\mathcal{A}',\mathcal{A}}\right| \le \|W_h(t)\|_{\mathcal{A}'}\left(\|\xi\cdot\nabla_x\varphi\|_{\mathcal{A}} + \|L_h(t)\|_{\mathcal{A}}\right)$$

and it only remains to prove that $\mathcal{F}_\eta L_h(t)$ is bounded in $L^1_yC^0_x$, uniformly with respect to $t\in[0,T]$ and h. Since $\Phi_h = \sigma_1 \star \int \sigma_2\psi_h\,\mathrm{d}z$,

$$\frac{1}{h}\left(\Phi_h(t,x+\frac{h}{2}y) - \Phi_h(t,x-\frac{h}{2}y)\right) = \frac{y}{h}\cdot\int_{-\frac{h}{2}}^{\frac{h}{2}}\nabla\sigma_1\star\left(\int_{\mathbb{R}^n}\sigma_2\psi_h(t)\,\mathrm{d}z\right)(x+sy)\,\mathrm{d}s$$

and we can estimate $\mathcal{F}_\eta L_h(t)$ as follows

$$\begin{aligned}\|\mathcal{F}_\eta L_h(t)\|_{L^1_yC^0_x} &\le \|y\mathcal{F}_\xi\varphi\|_{L^1_yC^0_x}\left\|\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\nabla\sigma_1\star\left(\int_{\mathbb{R}^n}\sigma_2\psi_h(t)\,\mathrm{d}z\right)(x+sy)\,\mathrm{d}s\right\|_{L^\infty_{x,y}}\\ &\le \|y\mathcal{F}_\xi\varphi\|_{L^1_yC^0_x}\left\|\nabla\sigma_1\star\left(\int_{\mathbb{R}^n}\sigma_2\psi_h(t)\,\mathrm{d}z\right)\right\|_{L^\infty_x}.\end{aligned}$$

The following estimate coupled with (51c) and Remark B.1 allows us to conclude

$$\left\|\nabla\sigma_1\star\left(\int_{\mathbb{R}^n}\sigma_2\psi_h(t)\,\mathrm{d}z\right)\right\|_{L^\infty_x} \le \|\nabla\sigma_1\|_{L^2_x}\|\sigma_2\|_{L^{p_0'}_z}\|\psi_h\|_{L^2_xL^\infty_tL^{p_0}_z}.$$

Step 6: Equation on μ. For any $\varphi\in\mathcal{B}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle W_h(t),\varphi\rangle_{\mathcal{B}',\mathcal{B}} = -\langle W_h(t),\xi\cdot\nabla_x\varphi\rangle_{\mathcal{B}',\mathcal{B}} + \langle W_h(t),L_h(t)\rangle_{\mathcal{B}',\mathcal{B}}.$$

The weak convergence of $(W_h)_{h>0}$ allows us to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle W_h(t),\varphi\rangle_{\mathcal{B}',\mathcal{B}} \underset{h\to0}{\longrightarrow} \frac{\mathrm{d}}{\mathrm{d}t}\langle \mu(t),\varphi\rangle_{\mathcal{B}',\mathcal{B}} \qquad \text{in } \mathcal{D}'(0,T),$$

and

$$\langle W_h(t),\xi\cdot\nabla_x\varphi\rangle_{\mathcal{B}',\mathcal{B}} \underset{h\to0}{\longrightarrow} \langle \mu(t),\xi\cdot\nabla_x\varphi\rangle_{\mathcal{B}',\mathcal{B}} \qquad \text{uniformly in time } (t\in[0,T]),$$

and it only remains to prove that $L_h(t)$ converges strongly in $\mathcal{A}$ (uniformly with respect to $t\in[0,T]$) to $\nabla_x\left(\sigma_1\star\int\sigma_2\psi(t)\,\mathrm{d}z\right)\cdot\nabla_\xi\varphi$, which is equivalent to prove the strong convergence of $\mathcal{F}_\xi L_h(t)$ to $iy\cdot(\nabla\sigma_1\star\int\sigma_2\psi(t)\,\mathrm{d}z)\mathcal{F}_\xi\varphi$ in $L^1_yC^0_x$. For that purpose we decompose the difference of these two terms as follows

$$\begin{aligned}&\mathcal{F}_\xi L_h(t,x,y) - iy\cdot\left(\int_{\mathbb{R}^d}\nabla\sigma_1(x-\bar{x})\left[\int\sigma_2(z)\psi(t,\bar{x},z)\,\mathrm{d}z\right]\mathrm{d}\bar{x}\right)\mathcal{F}_\xi\varphi(x,y)\\ &= iy\cdot\left(\int_{\mathbb{R}^d}\nabla\sigma_1(x-\bar{x})\left[\int\sigma_2(z)(\psi(t,\bar{x},z)-\psi_h(t,\bar{x},z))\,\mathrm{d}z\right]\mathrm{d}\bar{x}\right)\mathcal{F}_\xi\varphi(x,y)\\ &+iy\cdot\left(\int_{\mathbb{R}^d}\frac{1}{h}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}}\nabla\sigma_1(x-\bar{x})-\nabla\sigma_1(x+sy-\bar{x})\,\mathrm{d}s\right]\left[\int\sigma_2(z)\psi_h(t,\bar{x},z)\,\mathrm{d}z\right]\mathrm{d}\bar{x}\right)\mathcal{F}_\xi\varphi(x,y)\\ &= \mathrm{I}(t,x,y)+\mathrm{II}(t,x,y).\end{aligned}$$

We estimate the first term as follows (where the support of $\mathcal{F}_\xi\varphi$ is supposed to be included in the compact $K_1 \times K_2$)

$$\|\mathrm{I}(t)\|_{L^1_y C^0_x} \leq \|y\mathcal{F}_\xi\varphi\|_{L^1_y C^0_x} \sup_{x\in K_1} |\nabla\sigma_1 \star (\sigma_2(\psi(t)-\psi_h(t)))(x)|$$

and the weak convergence of $(\psi_h)_{h>0}$ insures us that for every $x \in K_1$

$$\begin{aligned}&\nabla\sigma_1 \star (\sigma_2(\psi(t)-\psi_h(t)))(x)\\ &\qquad = \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\zeta|\, \nabla\sigma_1(x-\bar{x})\frac{\hat{\sigma}_2(\zeta)}{|\zeta|^2}\, |\zeta| \overline{\left(\hat{\psi}(t,\bar{x},\zeta)-\hat{\psi}_h(t,\bar{x},\zeta)\right)}\, \mathrm{d}\bar{x}\, \mathrm{d}\zeta \underset{h\to 0}{\longrightarrow} 0.\end{aligned}$$

This convergence is not *a priori* uniform in $x \in K_1$. Nevertheless, we can combine the fact that $\psi(t)-\psi_h(t)$ is uniformly bounded with respect to t and h in $L^2_x\dot{H}^1_z$, K_1 is compact and the application

$$x \in \mathbb{R}^d \longmapsto \left((\bar{x},z)\mapsto \nabla\sigma_1(x-\bar{x})\mathcal{F}^{-1}_\zeta(\hat{\sigma}_2(\zeta)/|\zeta|^2)(z)\right) \in L^2_x\dot{H}^1_z$$

is continuous, to prove that the convergence is indeed uniform in x. For the second term, the estimate

$$\begin{aligned}\|\mathrm{II}(t)\|_{L^1_yC^0_x} \;&\leq\; \|y\mathcal{F}_\xi\varphi\|_{L^1_yC^0_x}\,\|\sigma_2\|_{L^{p_0'}_z}\,\|\psi_h\|_{L^2_xL^\infty_tL^{p_0}_z}\\ &\qquad\times \sup_{\substack{x\in K_1\\ y\in K_2}}\left(\int_{\mathbb{R}^d}\frac{1}{h^2}\left|\int_{-\frac{h}{2}}^{\frac{h}{2}} \nabla\sigma_1(x-\bar{x})-\nabla\sigma_1(x+sy-\bar{x})\,\mathrm{d}s\right|^2\right)^{1/2}\\ &= \|y\mathcal{F}_\xi\varphi\|_{L^1_yC^0_x}\,\|\sigma_2\|_{L^{p_0'}_z}\,\|\psi_h\|_{L^2_xL^\infty_tL^{p_0}_z}\sup_{y\in K_2}\left(\int_{\mathbb{R}^d}\frac{1}{h^2}\left|\int_{-\frac{h}{2}}^{\frac{h}{2}}\nabla\sigma_1(x)-\nabla\sigma_1(x+sy)\,\mathrm{d}s\right|^2\mathrm{d}x\right)^{1/2}\end{aligned}$$

coupled with the regularity and the compactness of the support of $\nabla\sigma_1$ and the uniform boundedness with respect to h of $\|\psi_h\|_{L^2_xL^\infty_tL^{p_0}_z}$, allows us to conclude that $\|\mathrm{II}(t)\|_{L^1_yC^0_x} \to 0$ when $h \to 0$.

Step 7: Final details. To conclude the proof it remains to justify that in fact the limit μ of the sequence $(W_h)_{h>0}$ defines an element of $C^0([0,T],\mathcal{M}_+ - w\star)$ and that the sequence $(|u_h|^2)_{h>0}$ converges in $C^0([0,T],\mathcal{M}(\mathbb{R}^d_x) - w\star)$ to $\rho = \int \mathrm{d}\mu(\xi)$. The first point comes from the study of the Husimi transform of u_h:

$$\widetilde{W}_h(t) = W_h(t) \star \frac{e^{-(|x|^2+|\xi|^2)/h}}{(\pi h)^d}.$$

One can prove that, for every time $t \in [0,T]$, $\widetilde{W}_h(t)$ is non negative and the sequence $(\widetilde{W}_h(t))_{h>0}$ is bounded in $L^1_xL^1_\xi$. This allows us to conclude that, up to a sub-sequence, $\widetilde{W}_h(t)$ converges weakly in the sense of measures to a certain $\tilde{\mu}(t) \in \mathcal{M}_+$ and it is then possible to prove that indeed $\mu(t) = \tilde{\mu}(t)$. We refer the reader to [26, Section III] for details. However it is not possible yet to conclude that μ is an element of $C^0([0,T],\mathcal{M}-w\star)$. In the previous argument each sub-sequence depends on t (then it is not possible to apply a diagonal argument) and we have no information about the time continuity. The missing step can be obtained by slightly modifying the compactness argument in **Step 5**, in order to obtain the compactness of the sequence $(\widetilde{W}_h)_{h>0}$ in $C^0([0,T],\mathcal{M}-w\star)$, and conclude that, up to a sub-sequence, $(\widetilde{W}_h)_{h>0}$ converges in $C^0([0,T],\mathcal{M}-w\star)$ to $\tilde{\mu} \in C^0([0,T],\mathcal{M}-w\star)$. We eventually obtain that $\mu = \tilde{\mu} \in C^0([0,T],\mathcal{M}-w\star)$.

Finally, we make use of the results in the [26, Section III] which tell us that if the sequence $(h^{-d}|\hat{u}_h(t,h^{-1}\xi)|^2)_{h>0}$ is tightly relatively compact, then $(|u_h(t)|^2)$ converges weakly in the sense of measures to $\rho(t) = \int \mathrm{d}\tilde{\mu}(t,\xi) = \int \mathrm{d}\mu(t,\xi)$. Moreover, we already know that $(\widetilde{W}_h)_{h>0}$ converges in $C^0([0,T], \mathcal{M} - w\star)$ to $\tilde{\mu}$, so that if $(h^{-d}|\hat{u}_h(t,h^{-1}\xi)|^2)_{h>0}$ is tightly relatively compact, *uniformly in time*, then the proof [26, Theorem III.1 point 3] can be revisited in order to obtain that $(|u_h|^2)_{h>0}$ converges in $C^0([0,T], \mathcal{M}(\mathbb{R}^d) - w\star)$ to $\rho = \int \mathrm{d}\tilde{\mu}(\xi) = \int \mathrm{d}\mu(\xi) \in C^0([0,T], \mathcal{M}(\mathbb{R}^d) - w\star)$.

Let us conclude the proof by proving that the sequence $(h^{-d}|\hat{u}_h(t,h^{-1}\xi|^2)_{h>0}$ is tightly relatively compact uniformly in time, which can be cast as

$$\sup_{t\geq 0}\sup_{h>0} \frac{1}{h^d}\int_{|\xi|\geq R} \left|\hat{u}_h(t,h^{-1}\xi)\right|^2 \mathrm{d}\xi \underset{R\to\infty}{\longrightarrow} 0.$$

Remark B.1, insures the existence of a constant $C > 0$, independent of $h > 0$ and $t \in [0,T]$, such that $h^2\|\nabla_x u_h(t)\|^2_{L^2_x} \leq C$. Then a direct computation shows that

$$\begin{aligned} h^2\int_{\mathbb{R}^d} |\nabla_x u_h(t,x)|^2 \,\mathrm{d}x &= h^2\int_{\mathbb{R}^d} |\xi|^2\,|\hat{u}_h(t,\xi)|^2 \,\mathrm{d}\xi \\ &= \frac{1}{h^d}\int_{\mathbb{R}^d} |\xi|^2 \left|\hat{u}_h(t,h^{-1}\xi)\right|^2 \mathrm{d}\xi \geq \frac{1}{h^d}\int_{|\xi|\geq R} R^2\left|\hat{u}_h(t,h^{-1}\xi)\right|^2 \mathrm{d}\xi, \end{aligned}$$

and we eventually obtain

$$\sup_{t\geq 0}\sup_{h>0} \frac{1}{h^d}\int_{|\xi|\geq R} \left|\hat{u}_h(t,h^{-1}\xi)\right|^2 \mathrm{d}\xi \leq \frac{C}{R^2}.$$

■

Acknowledgements

We are gratefully indebted to Stephan De Bièvre for many motivating discussions and warm encouragements. We also thank Enno Lenzmann for useful hints and kind advices. Finally, we thank David Chiron and one of the anonymous reviewer who both indicated relevant improvements of the arguments.

References

[1] R. Alonso, T. Goudon, and A. Vavasseur. Damping of particles interacting with a vibrating medium. *Ann. IHP. Anal. Non-Linéaire*, 34(7):1727–1758, 2017.

[2] F. Bolley, D. Cordero-Erausquin, Y. Fujita, I. Gentil, and A. Guillin. New sharp Gagliardo-Nirenberg-Sobolev inequalities and an improved Borell-Brascamp-Lieb inequality. *Int. Math. Res. Notices*, 10:3042–3083, 2020.

[3] L. Bruneau and S. De Bièvre. A Hamiltonian model for linear friction in a homogeneous medium. *Comm. Math. Phys.*, 229(3):511–542, 2002.

[4] A. O. Caldeira and A. J. Leggett. Quantum tunnelling in a dissipative system. *Ann. Phys.*, 149:374–456, 1983.

[5] T. Cazenave. *Semilinear Schrödinger Equations*, volume 10 of *Lecture Notes CIMS*. AMS, 2003.

[6] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.*, 85(4):549–561, 1982.

[7] P. D'Avenia and M. Squassina. Soliton dynamics for the Schrödinger-Newton system. *Math. Models Methods Appl. Sc.*, 24(03):553–572, 2014.

[8] S. De Bièvre, T. Goudon, and A. Vavasseur. Particles interacting with a vibrating medium: existence of solutions and convergence to the Vlasov–Poisson system. *SIAM J. Math. Anal.*, 48(6):3984–4020, 2016.

[9] S. De Bièvre, T. Goudon, and A. Vavasseur. Stability analysis of a Vlasov–Wave system describing particles interacting with their environmemt. *J. Diff. Eq.*, 264(12):7069–7093, 2018.

[10] M. del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl.*, 81(9):847–875, 2002.

[11] J. Ginibre and G. Velo. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal.*, 133(1):50–68, 1995.

[12] T. Goudon and A. Vavasseur. Mean field limit for particles interacting with a vibrating medium. *Annali Univ. Ferrara*, 62(2):231–273, 2016.

[13] T. Goudon and L. Vivion. Landau damping in dynamical Lorentz gases. Technical report, Univ. Côte d'Azur, Inria, CNRS, LJAD, 2019.

[14] T. Goudon and L. Vivion. Numerical investigation of Landau damping in dynamical Lorentz gases. *Phys. D.*, 403:132310, 2020.

[15] T. Goudon and L. Vivion. Numerical investigation of stability issues for quantum dissipative systems. Technical report, Université Côte d'Azur, Inria, CNRS, LJAD, 2020.

[16] M. Keel and T. Tao. Endpoint Strichartz estimates. *American J. of Math.*, 120:955–980, 1998.

[17] H. Kikuchi and M. Ohta. Stability of standing waves for the Klein-Gordon-Schrödinger system. *J. Math. Anal. Appl.*, 365:109–114, 2010.

[18] M. K. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. *Arch. Rational Mech. Anal.*, 105:243–266, 09 1989.

[19] E. Lenzmann. Uniqueness of ground states for pseudo-relativistic Hartree equations. *Anal. PDE*, 2:1–27, 01 2009.

[20] E. H. Lieb. Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Studies in Applied Mathematics*, 57(2):93–105, 1977.

[21] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. AMS, 2001. (2nd. edition).

[22] P.-L. Lions. Symétrie et compacité dans les espaces de Sobolev. *J. Funct. Anal.*, 49:315–334, 1982.

[23] P.-L. Lions. The concentration-compactness principle in the calculus of variations. the locally compact case, part 1. *Ann. IHP., Non Lin. Anal.*, 1(2):109–145, 1984.

[24] P.-L. Lions. The concentration-compactness principle in the calculus of variations. the locally compact case, part 2. *Ann. IHP., Non Lin. Anal.*, 1(2):223–283, 1984.

[25] P. L. Lions. *Mathematical topics in fluid mechanics; Volume 1: Incompressible models.* Clarendon Press, Oxford, 1996.

[26] P.-L. Lions and T. Paul. Sur les mesures de Wigner. *Revista Matemática Iberoamericana*, 9(3):553–618, 1993.

[27] P.L. Lions. The Choquard equation and related questions. *Nonlinear Analysis: Theory, Methods and Applications*, 4(6):1063–1072, 1980.

[28] L. Ma and L. Zhao. Classification of positive solitary solutions of the nonlinear Choquard equation. *Arch. Rational Mech. Anal.*, 195:455–467, 2010.

[29] Y. Martel. Notes on the interaction of solitary waves for NLS. `https://yvan_martel.wiki.math.cnrs.fr/uploads/Bonn2017.pdf`, 2017. Lectures notes for a course given in the summer school "Dispersive Equations, Solitons, and Blow-up" in September 2017 at the Hausdorff Center for Mathematics in Bonn.

[30] Y. Martel and F. Merle. Asymptotic stability of solitons for subcritical generalized KdV equations. *Arch. Rational Mech. Anal.*, 157:219–254, 2001.

[31] K. McLeod. Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$. II. *Trans. AMS*, 339:495–505, 02 1993.

[32] F. Merle. Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. *Duke Math. J.*, 69(2):427–454, 1993.

[33] L. Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa, Classe di Scienze*, 13(2):115–162, 1959.

[34] P. Raphaël. Séminaire Bourbaki : concentration compacité à la Kenig-Merle. In *Astérisque*, volume 352, pages 121–146. Soc. Math. France, 2013.

[35] E.M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, 1970.

[36] W. Strauss. Existence of solitary waves in higher dimensions. *Commun. Math. Phys.*, 55:149–162, 1977.

[37] M. E. Taylor. *Measure theory and integration*, volume 76 of *Graduate Studies in Math.* AMS, 2006.

[38] A. Vavasseur. *Some models of particles interacting with their environment.* PhD thesis, University Nice Sophia Antipolis, 2016.

[39] L. Vivion. *Particules classiques et quantiques en interaction avec leur environnement : analyse de stabilité et problèmes asymptotiques.* PhD thesis, Univ. Côte d'Azur, 2020.

[40] M. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16(3):472–491, 1985.

[41] M. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39:51–67, 01 1986.

[42] M. I. Weinstein. Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.*, 87(4):567–576, 1982.

[43] G. Zhang and N. Song. Travelling solitary waves for boson stars. *El. J. Diff. Eq.*, 2019:73: 1–12, 2019.